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FACTORIZATION AND REDUCTION METHODS FOR OPTIMAL CONTROL OF DISTRIBUTED PARAMETER SYSTEMS

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ABSTRACT

A Chandrasekhar-type factorization method is applied to the linear-quadratic optimal control problem for distributed parameter systems. An aeroelastic control problem is used as a model example to demonstrate that if computationally efficient algorithms, such as those of Chandrasekhar-type, are combined with the special structure often available to a particular problem, then an abstract approximation theory developed for distributed parameter control theory becomes a viable method of solution. A numerical scheme based on averaging approximations is applied to hereditary control problems. Numerical examples are given.

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I. INTRODUCTION

The problem of developing computational algorithms for distributed parameter control systems has been the subject of a large number of recent papers. These articles run the gamut from very abstract papers dealing with general approximation theory for infinite dimensional systems to explicit numerical algorithms derived expressly for a particular type of system and/or application. In any investigation of approximation schemes for such systems, we believe that it is important to keep in mind the ultimate goal of the approximation. For example, an approximation scheme that produces excellent results if used for parameter estimation might be inappropriate for computing feedback gain operators. The point at which approximations are introduced into the analysis is something that also varies with the problem and with the particular approach used to analyze the algorithm.

In this paper we concentrate on the linear quadratic optimal control problem for certain distributed parameter systems. We employ the approximation theory developed by Gibson [25] to formulate and analyze fast computational algorithms for approximating optimal feedback gain operators. These methods are based on factorization schemes of Chandrasekhar type.

A primary objective of this paper is to show that if one combines a computationally efficient algorithm such as a Chandrasekhar type method with the special structure of often available in "real problems," then many practical problems can be attacked using distributed parameter control theory and sound computational techniques. As noted in Casti and Ljung [15], this has been considered a major stumbling block between theoretical results for distributed parameter systems and their application to practical problems.

Our approach is based on an approximation theory specifically developed for approximating infinite dimensional control systems. Although the paper does contain some theoretical results on the existence, uniqueness, and smoothness of strong solutions to Chandrasekhar integral equations, we feel that the most significant aspects of this paper lie in the numerical results and applications. However, such theoretical results (especially smoothness) play an important role in the analysis of numerical schemes for direct integration of these equations (see Sorine [42]). These results have also been used to establish differentiability of strong solutions to operator Riccati differential equations (see [27], [44]).

Previous authors have studied infinite dimensional Chandrasekhar equations in connection with quadratic control problems [8], [15], [16], [44], and many of these authors have suggested that such algorithms should lead to very efficient computational techniques. However, except for a few examples [12], [15], [16], [40], very few numerical results have appeared in the open literature that substantiate these claims. Therefore, we have included a number of numerical examples to illustrate the computational aspects of these algorithms.

In Section 2 we present an aeroelastic control problem and briefly outline the derivation of the model. The control problem is then formulated as an infinite dimensional linear quadratic control problem. We introduce this model problem in order to provide an example to motivate our work and to check our computational algorithms. Section 3 is devoted to the development of the Chandrasekhar equations and computational algorithms for a general distributed parameter control problem. This section contains the statement of the major theoretical results. Proofs of these results are given in the Appendix.

In Section 4 we apply the general ideas developed in Section 3 to control problems governed by retarded functional differential equations. We restricted our attention to these problems for two reasons. First, except for some minor extensions, Gibson's approximation results [25] can be directly applied to obtain convergence of the Chandrasekhar algorithm. Secondly, we have considerable numerical experience with this class of problems, which allows us to compare the efficiency of a number of different numerical algorithms. We begin Section 4 by summarizing the basic results (see [2], [3], [25]) concerning the averaging approximation scheme for control problems governed by retarded functional differential equations. We also consider a "reduced averaging scheme" that takes advantage of the special structure that occurs in many problems and which often leads to considerable computational savings. Finally, we combine the Chandrasekhar algorithm with the reduced approximation scheme to produce a convergent and computationally efficient algorithm for approximating gain operators. The infinite time problem is also discussed.

In Section 5 we present a number of numerical examples to illustrate the computational aspects of the theoretical results. These examples illustrate the potential applicability of these factorization methods for very large-scale control problems. As noted above proofs will be given in the Appendix.

The notation used in this paper is fairly standard. Given two real Hilbert spaces X and Y, $\mathcal{L}(X,Y)$ shall denote the space of all bounded linear operators $\mathcal{B}:X \to Y$ with the usual operator norm $\|\mathcal{B}\|$. The innerproduct and the norm of the space X shall be denoted by $\langle\cdot,\cdot\rangle_X$ and $\|\cdot\|_X$, respectively; subscripts will be dropped if it is clear from the context which space is intended. The adjoint of a closed linear operator

$A:\mathcal{D}(A)\subseteq X \to Y$ shall be denoted by A^*. The resolvent set of A is denoted by $\rho(A)$. Finally, for an interval (a,b) of the real line, $L_2(a,b;X)$ denotes the set of L_2-integrable functions whose values lie in the Hilbert space X. If the Hilbert space is understood from the context, $L_2(a,b)$ shall be used.

II. AN AEROELASTIC CONTROL PROBLEM

In order to provide the reader with some concrete example and to motivate our work, we present a brief description of a problem that we shall use to test the computational algorithms. A more detailed derivation of the model may be found in [10].

Consider the problem of controlling the aeroelastic structure (i.e., the typical section) shown in Figure 2.1. The airfoil is placed in a flow field with undisturbed stream velocity U and allowed to plunge and pitch in the flow.

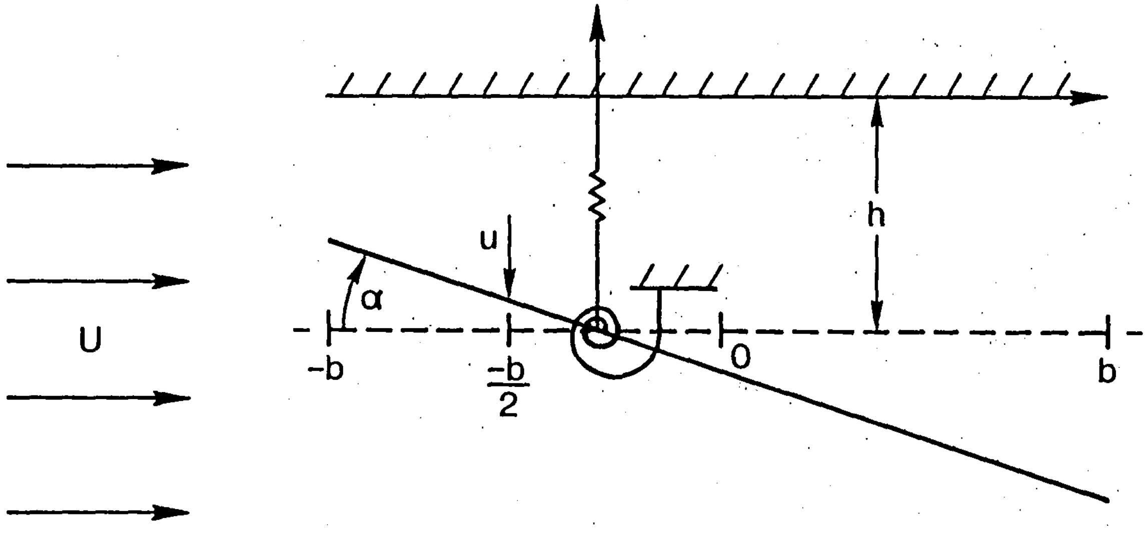

Figure 2.1

Let $h(t)$ denote the plunge and $\alpha(t)$ the pitch of the airfoil at time t. The equations of motion can be written in the form

$$M_s \ddot{y}(t) + K_s y(t) = F(t), \tag{2.1}$$

where $y(t) = (h(t), \alpha(t))^T$ and F contains the aerodynamic and applied loads on the airfoil. In particular

$$F(t) = \begin{bmatrix} L(t) + u(t) \\ M_\alpha(t) \end{bmatrix}, \tag{2.2}$$

where $L(t)$ and $M_\alpha(t)$ are the aerodynamic loads corresponding to the total wing lift per unit depth and total moment about the 1/4 chord per unit depth, respectively. For the airfoil considered here, it follows that (see [9], [10], [46])

$$M_\alpha(t) = \pi\rho b^3[\ddot{h}(t)/2 + 3b\ddot{\alpha}(t)/8 + U\dot{\alpha}(t)] \tag{2.3}$$

and

$$L(t) = \pi\rho b^2[\ddot{h}(t) + b\ddot{\alpha}(t) + U\dot{\alpha}(t)] + (2\pi\rho U b)D(t), \tag{2.4}$$

where ρ is the density of air, b is the semichord length and $D(t)$ is the "Duhamel integral." In particular, $D(t)$ is given by

$$D(t) = \int_0^t \Phi\left(\frac{U}{b}(t - s)\right)\dot{Q}(s)ds \tag{2.5}$$

where $\Phi(Ut/b)$ is the Wagner function (see [9]) and

$$Q(t) = \int_0^t [\ddot{h}(s) + b\ddot{\alpha}(s) + U\dot{\alpha}(s)]ds. \tag{2.6}$$

In order to obtain a state space model that is suitable for control, it is necessary to provide a useful representation for $D(t)$. Clearly, the key to this problem is the Wagner function Φ. It is possible to show that (see [11])

$$D(t) = Q(t) - g(t) \tag{2.7}$$

where $g(t)$ is the output to a hereditary system (i.e., a functional differential equation) with input $\dot{Q}(t)$. In particular, it follows that

$$\Phi(Ut/b) = 1 - W(t) \tag{2.8}$$

and

$$W(t) = \mathcal{C} e^{\mathcal{A} t} \mathcal{B} \tag{2.9}$$

is the weighting pattern for a hereditary control system. Almost all approaches to modeling aeroelastic structures can be reduced to some scheme for approximating $W(t)$. For example, R. T. Jones (see [30]) used the two-term exponential function

$$\tilde{W}(t) = \alpha_1 e^{-\beta_1 t} + \alpha_2 e^{-\beta_2 t} \tag{2.10}$$

with $\alpha_1 = 0.165$, $\alpha_2 = 0.335$, $\beta_1 = 0.0455$ and $\beta_2 = 0.3$. If $\tilde{W}(t)$ is substituted into (2.5), then $D(t)$ is approximated by

$$\tilde{D}(t) = Q(t) - \tilde{g}(t) = Q(t) - [\alpha_1 \tilde{x}_1(t) + \alpha_2 \tilde{x}_2(t)], \tag{2.11}$$

where $\tilde{x}_1(t)$, $\tilde{x}_2(t)$ satisfy the second-order ordinary differential equation

$$\frac{d}{dt}\begin{bmatrix}\tilde{x}_1(t)\\ \tilde{x}_2(t)\end{bmatrix} = \begin{bmatrix}-\beta_1 & 0\\ 0 & -\beta_2\end{bmatrix}\begin{bmatrix}\tilde{x}_1(t)\\ \tilde{x}_2(t)\end{bmatrix} + \begin{bmatrix}1\\ 1\end{bmatrix}\dot{Q}(t). \tag{2.12}$$

In particular, $\tilde{g}(t)$ is the output of the system (2.12) with input $\dot{Q}(t)$ and the Jones approximation is equivalent to approximating the weighting pattern $W(t)$ by

$$\tilde{W}(t) = [\alpha_1\ \alpha_2]e^{\begin{bmatrix}-\beta_1 & 0\\ 0 & -\beta_2\end{bmatrix}t}\begin{bmatrix}1\\ 1\end{bmatrix}. \tag{2.13}$$

As indicated above, one can show that the model should include hereditary terms (see [11]). We shall consider the simplest possible model of this nature. This shall be accomplished by replacing (2.12) with the delay-differential equation

$$\dot{x}^d(t) = g_1\, x^d(t) + g_2\, x^d(t - r) + \dot{Q}(t) \tag{2.14}$$

and approximate $D(t)$ by

$$D^d(t) = Q(t) - g^d(t) = Q(t) - c_1\, x^d(t) \tag{2.15}$$

with system parameters g_1, q_2, c_1, and delay $r > 0$ (normally these parameters must be identified using a parameter estimation scheme). The problem of estimating the time delay r has been considered in other papers [4], [10]. The delay is clearly dependent on the chord length b and the undisturbed stream velocity U. We have found that for problems considered below a reasonable asymptotic "first estimate" of r is $r \approx \frac{kb}{U}$ where $k = 10$. In this model, $x^d(t)$ represents a generalized aerodynamic "lag state." To complete the model we augment equations (2.1) - (2.4) with equation (2.14) and replace $D(t)$ with the approximation $D^d(t)$ defined by (2.15). Let $x(t)$ denote the five-dimensional vector

(2.16) $$x(t) = \left(\dot{h}(t), \dot{\alpha}(t), h(t), \alpha(t), x^d(t)\right)^T.$$

The basic model becomes

(2.17) $$E\dot{x}(t) = F_0\, x(t) + F_1\, x(t - r) + Gu(t)$$

with initial data

(2.18) $$x(0) = \eta \in \mathbb{R}^5;\ x^d(s) = \phi(s) \in L_2(-r,0;\mathbb{R})$$

and output

(2.19) $$y(t) = Cx(t) \in \mathbb{R}^p.$$

The control problem we consider is to find $u^*:[0,T] \to \mathbb{R}^1$ that minimizes

(2.20) $$J(u) = \int_0^T [\|y(s)\|^2 + Ru^2(s)]ds$$

where $R > 0$ and $y(t)$ is the output to the delay-differential system (2.17) - (2.19).

It is important to note that only the aerodynamic lag state $x^d(t)$ involves a time delay (i.e., the first four columns of F_1 are zero). This observation will be important when we consider the computational algorithms in Section 5 below. The matrices E, F_0, F_1, G, and C are given in the technical report [10]. Parameters needed to complete the modeling process can be obtained by applying the parameter identification methods presented in [10] to experimental wind tunnel data. A specific example of this process is given in [10].

There are a number of state space formulations for this problem. The approach we take here is a slight variation of the "standard" formulation given in references [9], [24], [46]. Let $A_0 = E^{-1} F_0$, $A_1 = E^{-1} F_1$, $B = E^{-1} G$ and note that A_1 has the form

(2.21) $$A_1 = \left[0 \;\vdots\; A_1^d \right],$$

where A_1^d is a 5×1 (i.e., column) vector. We choose $\mathcal{H} = \mathbb{R}^5 \times L_2(-r,0;\mathbb{R})$ as our state space and let $(\eta,\phi(\cdot)) = (\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \phi(\cdot))$ denote a typical element in $\mathcal{H}$. Define the linear operator $\mathcal{A}$ on $\mathcal{H}$

(2.22) $$\mathcal{D}(\mathcal{A}) = \{(\eta, \phi(\cdot)) | \phi(\cdot) \in W^{1,2}(-r,0;\mathbb{R}),\ \phi(0) = \eta_5\}$$

by

(2.23) $$\mathcal{A}(\eta, \phi(\cdot)) = (A_0 \eta + A_1^d \phi(-r), \dot{\phi}(\cdot)).$$

Moreover, we define $\mathcal{B}: \mathbb{R} \to \mathcal{H}$ and $\mathcal{V}: \mathcal{H} \to \mathbb{R}^p$ by

(2.24) $$\mathcal{B} u = (Bu, \theta)$$

and

$$V(\eta,\phi(\cdot)) = C\eta, \tag{2.25}$$

respectively. The delay-differential system (2.17) - (2.19) can be realized (see [10]) as the system

$$\dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}u(t) \tag{2.26}$$

$$z(0) = z_0 = (\eta,\phi(\cdot)) \tag{2.27}$$

$$y(t) = Vz(t). \tag{2.28}$$

Moreover, the optimal control problem is equivalent to finding $u^*:[0,T] \to \mathbb{R}^1$ that minimizes

$$J(u) = \int_0^T [\|y(s)\|^2 + Ru^2(s)]ds, \tag{2.29}$$

where $y(t)$ is the output to (2.26) - (2.28).

The problem defined by (2.26) - (2.29) will be used to test the numerical schemes described in Section 4. However, it also serves to motivate the theoretical developments presented below.

III. THE CHANDRASEKHAR ALGORITHM

In this section our attention is focused on the time-invariant infinite dimensional linear regulator problem. For a study of control problems governed by a general evolution equation we refer the interested reader to Curtain and Pritchard [18], Gibson [26], and Datko [19]. The evolution processes usually arise from control of systems governed by partial differential equations (see Lions [35] and Lukes and Russell [36]), or functional differential equations (see Delfour and Mitter [23] and Manitius [38]). The approach we follow is similar to the development of the optimal control problem in Gibson [25]. The importance and potential usefulness of Chandrasekhar equations have been known for some time. There are a number of papers that discuss the application of these equations to finite dimensional linear quadratic optimal control problems (i.e., see [13], [14], [31], [22]). However, the derivation of these equations relied upon being able to twice differentiate the solution to a matrix Riccati differential equation. In an infinite dimensional setting there are a number of questions concerning the existence of these derivations (in a strong sense) that limit the usefulness of this approach for infinite dimensional systems. The derivation of the infinite dimensional equations in [45] by Tung is purely formal. The differentiations used in his derivation were not justified. In [8] and [15] the authors used a Lions-type setting and derived a set of Chandrasekhar differential equations satisfied in a distributional sense. We intend to give an alternate derivation which will lead to a set of Chandrasekhar integral equations that have unique strong solutions. Moreover, under fairly weak assumptions it can be shown that the solution to these integral equations is strongly differentiable (a result that can be applied to establish smoothness properties of solutions to Riccati operator differential equations [27]).

In [44], Sorine established that the gain operator satisfies a set of Chandrasekhar equations if the underlying semigroup of the system is analytic. For systems governed by hyperbolic PDE′s or differential-delay equations (such as the aeroelastic system above) the associated semigroups are not analytic, and hence Sorine's results do not apply. In this section we give a set of Chandrasekhar equations in an integral form for a large class of optimal control problems governed by a general distributed parameter model. The results are established by an approximation technique and leads to computationally feasible methods. Because the proofs are technical and provide little insight into the structure of the problem, we have placed them in an appendix.

Let H and U be real Hilbert spaces. Throughout this section $T(t):H \to H$, $t \geq 0$, will denote a strongly continuous C_0-semigroup of bounded linear operators with infinitesimal generator A. We shall always assume that the following basic hypothesis holds.

H_0) i) The operators $B:U \to H$, $Q:H \to H$ and $R:U \to U$ are continuous linear operators.

ii) The operators Q and R are self-adjoint and non-negative and R satisfies $\|R\| \geq m > 0$.

iii) The operator Q can be factored into the form $Q = V^* V$ where $V:H \to \Lambda$ is a bounded linear operator and Λ is a Hilbert space.

The infinite dimensional linear quadratic (LQ) optimal control problem is to find a $u^* \in L_2(0,t_f;U)$ which minimizes

$$J(z,u) = \int_0^{t_f} [\langle Qz(s),z(s)\rangle + \langle Ru(s),u(s)\rangle]ds \tag{3.1}$$

where $z(t)$ is defined by

$$z(t) = T(t)z + \int_0^t \mathcal{B}T(t - s)\, u(s)d\sigma \tag{3.2}$$

for $0 \leq t \leq t_f$ and $z \in \mathcal{H}$. Although our discussion is mainly restricted to the control problem on finite intervals, we will have occasion to discuss its relationship to the infinite interval control problem. In this case the cost functional (3.1) becomes

$$J_\infty(z,u) = \int_0^\infty [\langle Qz(s),z(s)\rangle + \langle Ru(s),u(s)\rangle]ds. \tag{3.3}$$

Under the assumptions given above on Q and R, it is known that (see [18], [25] for example) there exists a unique $u^* \in L_2(0,t_f;\mathcal{U})$, (respectively $L_2(0,\overline{t};\mathcal{U})$ for each $\overline{t} > 0$) which minimizes $J(z,u)$ (respectively $J_\infty(z_0,u)$). The following characterization of this optimal control may be found in Gibson [25]. This characterization is discussed in detail in order to develop the notation we shall need for the derivation of the Chandrasekhar equations.

For $0 \leq s \leq t_f$ define the Hilbert spaces $\mathcal{H}_s$ and $\mathcal{U}_s$ by $\mathcal{H}_s = L_2(s,t_f;\mathcal{H})$ and $\mathcal{U}_s = L_2(s,t_f;\mathcal{U})$, respectively. Let the operators $T_s \in \mathcal{L}(\mathcal{H},\mathcal{H}_s)$, $\mathcal{T}_s \in \mathcal{L}(\mathcal{H}_s,\mathcal{H}_s)$, and $\mathcal{F}_s \in \mathcal{L}(\mathcal{H}_s,\mathcal{H})$ be defined by

$$(T_s\, z)(t) = T(t - s)z, \qquad z \in \mathcal{H},$$

$$(\mathcal{T}_s\, \phi)(t) = \int_s^t T(t - \eta)\phi(\eta)d\eta, \quad 0 \leq s \leq t \leq t_f, \quad \phi \in \mathcal{H}_s,$$

and

$$\mathcal{F}_s\, \phi = (\mathcal{T}_s\, \phi)(t_f),$$

respectively. Straightforward calculations imply that $\mathcal{T}_s^*$ is given by

$$(\mathcal{T}_s^* \psi)(t) = \int_t^{t_f} T^*(\eta - t)\psi(\eta)d\eta, \qquad \psi\epsilon\mathcal{H}_s$$

and that $\mathcal{F}_s^*$ has the form

$$(\mathcal{F}_s^* z)(t) = T^*(t_f - t)z, \qquad z\epsilon H.$$

These representations will be useful in the proofs of Lemmas 3.1 and 3.2 in the Appendix.

For the optimal control problem on the interval $[s,t_f]$ with $0 \leq s \leq t_f$, define the cost functional

$$J(s,z(s),u) = + \int_s^{t_f} (\langle Qz(\eta),z(\eta)\rangle + \langle Ru(\eta),u(\eta)\rangle)d\eta,$$

where $z(t)$ is defined by (3.2). In [25] Gibson has shown that the optimal control may be expressed in the form

$$u^*(t) = -(\hat{R}_s^{-1} \hat{\mathcal{B}}_s^*)(t)z^*(s), \qquad 0 \leq s \leq t, \tag{3.4}$$

for almost all t in $[0,t_f]$ where $\hat{R}_s$ and $\hat{\mathcal{B}}_s^*$ are defined by

$$\hat{R}_s = R + \mathcal{B}^* \mathcal{T}_s^* Q\mathcal{T}_s \tag{3.5}$$

and

$$\hat{\mathcal{B}}_s^* = \mathcal{B}^* \mathcal{T}_s^* Q\mathcal{T}_s, \tag{3.6}$$

respectively. Note that $\hat{R}_s \in L(U_s, U_s)$ and $\hat{B}_s^* \in L(H, U_s)$. Since $T(t)z = T(t - s)z^*(s)$ it follows directly from (3.2) and (3.4) that the optimal trajectory $z^*(t)$ has the representation

$$z^*(t) = T(t - s)z^*(s) - \int_s^t T(t - \eta)B(\hat{R}_s^{-1} \hat{B}_s^*)(\eta)z^*(s)d\eta, \tag{3.7}$$

for $0 \leq s \leq t \leq t_f$. If $S(t,s) \in L(H,H)$ is defined by

$$S(t,s)z = T(t - s)z - \int_s^t T(t - \eta)B(\hat{R}_s^{-1} \hat{B}_s^*)(\eta)zd\eta, \tag{3.8}$$

then (3.7) becomes

$$z^*(t) = S(t,s)z, \qquad 0 \leq s \leq t \leq t_f.$$

In [25], Gibson shows that $S(t,s)$ is the bounded perturbation of $T(t)$ by $-BR^{-1} B^* \Pi(t)$ where $\Pi(t)$ is defined by

$$\Pi(t)z = \int_t^{t_f} T^*(\eta - t)QS(\eta,t)zd\eta, \qquad z \in H, \tag{3.9}$$

Furthermore, he shows that the optimal control has the representation

$$u^*(t) = -R^{-1} B^* \Pi(t)z^*(t) = -K(t)z^*(t).$$

It has also been shown (see [18] and [25]) that $\Pi(t)$ is the unique self-adjoint solution to the two Riccati-type integral equations

$$\Pi(t)z = \int_t^{t_f} T^*(s - t)[Q - \Pi(s)BR^{-1} B^* \Pi(s)]T(s - t)zds \tag{3.10}$$

and

$$\Pi(t)z = \int_{t}^{t_f} S^*(s,t)[Q + \Pi(s)\mathcal{B}R^{-1}\mathcal{B}^* \Pi(s)]S(s,t)z\,ds. \tag{3.11}$$

A formal differentiation of either of these two equations yields a Riccati differential equation analogous to the case for the finite dimensional linear regulator problem. Such formal differentiations will be avoided in our derivation of the infinite dimensional Chandrasekhar equations.

As in the finite dimensional case, under appropriate stabilizability and detectability requirements, the solution of the infinite time problem may be viewed as the limit of the solution to the finite time problem as $t_f \to \infty$. Our only interest in this problem is to show that the Chandrasekhar equations may also lead to a computationally feasible method for computing solutions to these problems. Numerical examples will be given in Section 5 to illustrate this idea and to discuss some of its limitations.

Since our main objective is the development of computational algorithms for numerically calculating the feedback gain operator for the optimal control problem (3.1) - (3.2), we shall approach the development of the Chandrasekhar equations via approximation theory. This theory is based on Gibson's work [25], [26], and our presentation makes use of a slight extension of his results. Again, we want to emphasize that these theoretical extensions are minor and that the real advantage of our approach lies in the computational savings that come from employing the problem structure and Chandrasekhar methods. However, this extension will enable us to show that the gain operator $K(t)$ for the system (3.1) - (3.2) satisfies, in a strong sense, a set of Chandrasekhar integral equations. Moreover, examples will be given to show that approximation of the gain through the Chandrasekhar equations can be computationally more efficient than the usual method of solving a Riccati equation.

Let $\{T^N(t)\}_{t\geq 0}$ be a sequence of C_0-semigroups on H with corresponding infinitesimal generators $\{A^N\}$. We also assume that $B^N \epsilon L(U,H)$ and $Q^N \epsilon L(H,H)$ define sequences of operators with each Q^N self-adjoint and non-negative. The following hypothesis is needed.

H_1) i) There exist constants Γ_0, Γ_1, Γ_2 such that

$$\|T^N(t)\| \leq \Gamma_0$$

for all N and $t \epsilon [0,t_f]$ and

$$\|B^N\| \leq \Gamma_1, \qquad \|Q^N\| \leq \Gamma_2$$

for all N.

ii) The semigroups $T^N(t)$ and $[T^N(t)]^*$ converge strongly to $T(t)$ and $T^*(t)$, respectively and the convergence is uniform in t on $[0,t_f]$.

iii) The operators B^N, $[B^N]^*$ and Q^N converge strongly to B, B^* and Q, respectively.

It is important to note that condition H_1 - (ii) requires convergence of the semigroups and their adjoints. This is a critical assumption if one is interested in applying these results to non-self-adjoint problems.

The Nth approximate LQ optimal control problem is to find $[u^N]^*(t)$ which minimizes

$$(3.12)\qquad J^N(z^N,\ u^N) = \int_0^{t_f} [\langle Q^N z^N(s), z^N(s)\rangle + \langle Ru^N(s), u^N(s)\rangle]ds$$

where $z^N(t)$ is defined by

$$(3.13)\qquad z^N(t) = T^N(t)z(0) + \int_0^t T^N(t-s)\mathcal{B}^N u^N(s)ds$$

for $0 \leq t \leq t_s$. Under the above assumptions, Gibson [25] has shown that the optimal control $[u^N]^*$ converges in $L_2(0,t_f;\mathcal{U})$ to the optimal control u^* for the original problem. In particular, he proved that the Riccati operators $\Pi^N(t)$ converge strongly to $\Pi(t)$, and uniformly in t on compact t-intervals This yields strong convergence of the gains $K^N(t) \equiv R^{-1}\mathcal{B}^N\Pi^N(t)$ to $K(t) \equiv R^{-1}\mathcal{B}\Pi(t)$. If the control space is finite dimensional, the convergence of $K^N(t)$ is uniform in norm.

In order to establish analogous results for Chandrasekhar equations, one needs the following technical lemmas, the proofs of which appear in the Appendix.

LEMMA 3.1: *If* $\mathcal{H}_0$ *and* $\mathcal{H}_1$ *are satisfied, then* $\|\hat{\mathcal{B}}_s^N[\hat{R}_s^N]^{-1}(t)\|$ *is uniformly bounded in* N, s, *and* t *for* $0 \leq s \leq t \leq t_f$.

Gibson [25] established that conditions $\mathcal{H}_0$ and $\mathcal{H}_1$ are sufficient to ensure the strong convergence of $S^N(t,s)$ to $S(t,s)$. We make use of a slightly more general result.

LEMMA 3.2: If $\mathcal{H}_0$ and $\mathcal{H}_1$ hold, then for each $z \in \mathcal{H}$, $0 \leq s \leq t \leq t_f$

$$S^N(t,s)z \to S(t,s)z$$

and

$$[S^N(t,s)]^* z \to S^*(t,s)z,$$

where the convergence is uniform in s and t. In addition, $S^N(t,s)$ and $[S^N(t,s)]^*$ are uniformly bounded in N, s, t for $0 \leq s \leq t \leq t_f$.

In order to derive an infinite dimensional version of the Chandrasekhar equations, it is necessary to make an additional assumption on the approximating sequences. This condition essentially requires that each of the approximating LQ control problems (3.12) - (3.13) have optimal gain operators that satisfy a form of the Chandrasekhar differential equations.

$\mathcal{H}_2$) i) There exists a sequence of approximating LQ optimal control problems (3.12) - (3.13) satisfying $\mathcal{H}_1$.

ii) The operators Q^N can be factored into $Q^N = [V^N]^* V^N$ with $V^N \to V$ and $[V^N]^* \to V^*$ strongly.

iii) The optimal gain operator $K^N(t)$ for the problem (3.12) - (3.13) is strongly continuously differentiable and satisfies

$$\dot{K}^N(t)z = -R^{-1} \mathcal{B}^{N*}[L^N(t)]^* L^N(t), \qquad K^N(t_f) = 0 \tag{3.14}$$

$$\dot{L}^N(t)z = -L^N(t)[A^N - \mathcal{B}^N K^N(t)]z, \qquad L^N(t_f) = V^N \tag{3.15}$$

for $z \in \mathcal{H}$, where A^N is the infintesimal generator of $T^N(t)$ and $L^N(t) = V^N S^N(t_f,t)$.

Condition $\mathcal{H}_2$) - (iii) is always satisfied if each A^N is a bounded linear operator. In this case the semigroups $T^N(t)$ are differentiable, and the derivation of (3.14) - (3.15) proceeds as in the finite dimensional case. It should also be noted that conditions $\mathcal{H}_1$) - (iii) and $\mathcal{H}_2$) - (ii) are independent in that strong convergence of Q^N to Q does not imply $\mathcal{H}_2$) - (iii) and conversely that $\mathcal{H}_2$) - (iii) is not sufficient to imply strong convergence of Q^N to Q.

We may now state the fundamental existence result.

THEOREM 3.3: Suppose that conditions $\mathcal{H}_0$, $\mathcal{H}_1$ and $\mathcal{H}_2$ hold. If K(t) denotes the optimal gain operator for problem (3.1) - (3.2), then K(t) satisfies the system of Chandrasekhar integral equations

(3.16)
$$K(t)z = \int_t^{t_f} \mathcal{R}^{-1} \mathcal{B}^* L^*(s)L(s)z\,ds$$

(3.17)
$$L(t)z = VT(t_f - t)z - \int_t^{t_f} L(s)\mathcal{B}K(s)T(s - t)z\,ds$$

for all $z \in \mathcal{H}$ and $0 \leq t \leq t_f$, and $L(t)z = VS(t_f,t)z$. Moreover, the approximate gain operators $K^N(t)$ converge strongly to K(t). If $\mathcal{U}$ is finite dimensional, then

(3.19)
$$\lim_{N \to \infty} \|K^N(t) - K(t)\| = 0$$

and the convergence is uniform in t on $[0,t_f]$.

Theorem 3.3 does not imply that K(t) and L(t) are the only strongly continuous solutions to (3.16) - (3.17). The proof that these equations have unique strongly continuous solutions is non-trivial, but essential if we hope to use these equations as a basis for computational algorithms.

THEOREM 3.4: Assume that H_0, H_1 and H_2 hold. Then K(t) and L(t) = VS(t_f,t) are the unique strongly continuous solutions to the Chandrasekhar equations (3.16) - (3.17).

Some comments concerning the assumptions H_0, H_1 and H_2 are in order. Although these conditions may at first glance seem severe, they are in fact the properties one would like to have in a scheme that is to be used to numerically approximate (3.1) - (3.2). In this case the convergence properties are assured and the Chandrasekhar integral equations may be approximated directly in order to obtain the gain operator. The averaging approximation scheme which is discussed in Section 4 satisfies these conditions when applied to linear regulator problems governed by delay-differential equations. For distributed parameter systems, Lukes and Russell [36] state conditions which also satisfy these hypotheses. Their approximations are essentially eigenfunction expansions and apply to problems whose infinitesimal generators have normal extensions (i.e., heat and wave equations with the appropriate boundary conditions).

While our major concern is to illustrate that approximation of the gain operator through the Chandrasekhar equations may have computational advantages over approximation of the Riccati operator, it should be pointed out that the theory developed is not restricted to finite dimensional approximations. This led Kazufumi Ito (see [27]) to note that if condition H_0 was satisfied, then the Yosida approximations could be used to define approximating sequences that satisfy H_1 and H_2. In particular, let $A^N = NA(NI - A)^{-1}$ (for N sufficiently large so that $N \in \rho(A)$) and define $B^N = B$, $Q^N = Q$, and $V^N = V$ for each N. It can be shown that these approximating sequences satisfy H_1 and H_2 and hence one has the following theorem on the existence and uniqueness of Chandrasekhar integral equations.

THEOREM 3.5: If H_0 is satisfied, then $K(t)$ and $L(t) = VS(t_f,t)$ are the unique strongly continuous solutions to the Chandrasekhar equations (3.16) - (3.17).

Although the Yosida approximates provide a set of approximating sequences that satisfy H_1 and H_2, in general these approximations are not very useful for numerical schemes. For example, the A^N operators are infinite dimensional and not easily constructed. It is for this reason that we have stated the conditions that are needed to not only obtain existence and uniqueness but to also guarantee the convergence of the gain operators.

We turn now to the application of these ideas to a particular class of distributed parameter systems. In particular, we shall concentrate on control systems governed by functional differential equations.

IV HEREDITARY CONTROL PROBLEMS

In this section we consider the control of a linear hereditary system of the form

$$\begin{aligned} &\dot{x}(t) - Lx_t + B_0\, u(t), \qquad 0 \leq t \leq t_f \\ &x(0) = \eta, \qquad x_0 = \phi, \end{aligned} \tag{4.1}$$

where $\eta \in \mathbb{R}^n$, $\phi \in L_2(-r,0;\mathbb{R}^n)$, $u \in L_2(0,t_1;\mathbb{R}^m)$ for each $t_1 < \infty$, $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$, and $x_t = x(t+\theta)$ for $-r \leq \theta \leq 0$. The linear operator L has the form

$$Lx_t = \sum_{i=0}^{\nu} A_i\, x(t-h_i) + \int_{-r}^{0} D(\theta)x(t+\theta)d\theta,$$

where each $A_i \in \mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$, $0 = h_0 < h_1 < \cdots < h_\nu = r$, and $D \in L_2(-r,0;\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n))$. Let Q be a real, symmetric, nonnegative $n \times n$ matrix, and R a real, symmetric, positive $m \times m$ matrix. The optimal control problem is to find $u^*(t) \in L_2(0,t_f;\mathbb{R}^m)$ which minimizes

$$J(u) \int_0^{t_f} [\langle Qx(s),x(s)\rangle - \langle Ru(s),u(s)\rangle]ds \tag{4.2}$$

where $x(t)$ is the solution to (4.1) corresponding to $u(t)$.

It is well-known that (see [3] for example) the system (4.1) has a unique solution for each $u(t) \in L_2(0,t_f;\mathbb{R}^m)$ and $(\eta,\phi) \in \mathbb{R} \times L_2(-r,0;\mathbb{R}^n) \equiv Z$. In addition, the solution depends continuously on η, ϕ, and u. Under the above stated conditions on Q and R there exists a unique $u^* \in L_2(0,t_f;\mathbb{R}^m)$ which minimizes (4.2) subject to (4.1).

In recent years a standard technique for obtaining the solution to (4.1)-(4.2) has been to reformulate the system as an evolution equation on the Hilbert space Z. The solution to problem (4.1) - (4.2) is then approximated via the abstract formulation. Taking this approach, we will see that the dynamical system falls into the framework described in the previous sections and the optimal gain operator is thus characterized via the Chandrasekhar equations (3.16) - (3.17).

For $(\eta,\phi(\cdot))\in Z$, define the operator $T(t)\in\mathcal{L}(\mathcal{H},\mathcal{H})$ by

$$T(t)(\eta,\phi(\cdot)) = (x(t),x_t(\cdot)) \tag{4.3}$$

where $x(t)$ is the solution to the homogeneous version of (4.1), i.e., $u(t) \equiv 0$. It is well-known (see [3]) that $T(t)$ is a C_0-semigroup on Z with infinitesimal generator $\mathcal{A}$ characterized by

$$\mathcal{D}(\mathcal{A}) = \{(\eta,\phi(\cdot)) \mid \phi(\cdot)\in W^{1,2}(-r,0;\mathbb{R}^n),\ \eta = \phi(0)\} \tag{4.4}$$

and

$$\mathcal{A}(\eta,\phi(\cdot)) = (L\phi,\dot{\phi}(\cdot)) \tag{4.5}$$

for $(\eta,\phi(\cdot))\in\mathcal{D}(\mathcal{A})$. Moreover, the delay differential equation (4.1) is equivalent to the abstract evolution equation (AEE)

$$z(t) = T(t)(\eta,\phi(\cdot)) + \int_0^t T(t-s)\mathcal{B}u(s)ds \tag{4.6}$$

where $\mathcal{B}u = (B_0 u,0)$. In particular, the following result may be found in [4].

THEOREM 4.1: _Let_ $(\eta,\phi)\in Z$ _be given. If_ $x(t;u)$ _is the solution of (4.1) for_ $u\in L_2(0,t_f;\mathbb{R}^m)$, _then_ $z(t)$ _defined by (4.6) satisfies_ $z(t) = (x(t;u), x_t(\cdot;u))$, $t \geq 0$.

Define the operator $\tilde{Q}\in L(Z,Z)$ by

$$\tilde{Q} = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$$

where 0 denotes the zero operator on the appropriate spaces. The control problem (4.1) - (4.2) is equivalent to finding $u^*(\cdot)\in L_2(0,t_f;\mathbb{R}^m)$ which minimizes

$$\tilde{J} = \int_0^{t_f} [\langle\tilde{Q}z(s),z(s)\rangle + \langle Ru(s),u(s)\rangle]ds \tag{4.7}$$

where $z(t)$ is defined by (4.6). Since the problem defined by (4.6) - (4.7) falls into the framework above, the optimal control has the representation

$$u^*(t) = -R^{-1}\,\mathcal{B}^*\,\Pi(t)z^*(t) = -K(t)z^*(t) \tag{4.8}$$

where $\Pi(t)$ satisfies the Riccati integral equations (3.10) and (3.11) and $z^*(t)$ is the optimal trajectory. Due to the special structure of the state space $Z = \mathbb{R}^n \times L_2(-r,0;\mathbb{R}^n)$, $\Pi(t)$ may be expressed as the matrix of operators

$$\Pi(t) = \begin{bmatrix} \hat{\Pi}_{00}(t) & \hat{\Pi}_{01}(t) \\ \hat{\Pi}_{10}(t) & \hat{\Pi}_{11}(t) \end{bmatrix}, \quad 0 \leq t \leq t_f \tag{4.9}$$

where there exists a real, nonnegative, symmetric $n \times n$ matrix $\Pi_{00}(t)$, and $\hat{\Pi}_{00}(t)\eta = \Pi_{00}(t)\eta$, $\hat{\Pi}_{10}(t)$ is a bounded linear operator from $L_2(-r,0;\mathbb{R}^n)$, and can be represented by

$$[\hat{\Pi}_{10}(t)\eta](s) = \Pi_{10}(t,s)\eta$$

where $\Pi_{10}(t,s)$ is an $n \times n$ matrix function, $\hat{\Pi}_{01}(t)$ is the bounded linear operator from $L_2(-r,0;\mathbb{R}^n)$ into $\mathbb{R}^n$ given by

$$\hat{\Pi}_{01}(t)\phi = \int_{-r}^{0} \Pi_{10}^T(t;s)\phi(s)ds, \qquad \phi\epsilon L_2(-r,0;\mathbb{R}^n), \tag{4.10}$$

and $\hat{\Pi}_{11}(t)$ is a real, non-negative, self-adjoint operator on $L_2(-r,0;\mathbb{R}^n)$. Further properties of these operators may be found in [25], [33], and [35]. Combining (4.8) - (4.10), it follows that the optimal control may be written as

$$u^*(t) = -R^{-1} B_0^T[\Pi_{00}(t)x^*(t) + \int_{-r}^{0} \Pi_{10}^T(t,s)x_t^*(s)ds] \tag{4.11}$$

for $0 \leq t \leq t_f$. Therefore, the gain operator has the form

$$K(t)(\eta,\phi(\cdot)) = K_0(t)\eta + \int_{-r}^{0} K_1(t,s)\phi(s)ds \tag{4.12}$$

where $K_0(t) = R^{-1} B_0^{-1} \Pi_{00}(t)$ is an $m \times n$ matrix and $K_1(t,s) = R^{-1} B_0^T \Pi_{10}^T(t,s)$ is an $m \times n$ matrix valued function. For each $t \geq 0$ the function $s \to K_1(t,s)$ is called the functional gain.

It is important to point out that the Riccati operator maps an infinite dimensional space to itself while the gain operator maps an infinite dimensional space to a finite dimensional space. Note that the gain is the actual operator needed to produce the optimal control. Thus it should be advantageous to approximate $K(t)$ instead of $\Pi(t)$ since it involves approximations in one spatial variable only. This difference can be accounted for by examining equations (4.9) and (4.10). The infinite dimensionality in two spatial variables of $\Pi(t)$ is due to the operators $\hat{\Pi}_{11}(t)$ and $\hat{\Pi}_{10}(t)$. However, the operator $\hat{\Pi}_{11}(t)$ does not appear in the representation (4.11), and the information in $\hat{\Pi}_{10}(t)$ is contained in the operator $\hat{\Pi}_{01}(t)$. The Chandrasekhar equations exploit this reduced dimensionality, and for the particular approximation scheme below it will be seen that this reduction can be quite significant.

We turn now to a particular approximation scheme (the so-called AVE scheme [3], [4], [25]) for the simple delay differential equation

$$(4.13)\qquad \begin{aligned} &\dot{x}(t) = A_0\, x(t) + A_1\, x(t-r) + B_0\, u(t) \\ &x(0) = \eta, \quad x(s) = \phi(s), \quad -r \le s < 0. \end{aligned}$$

This scheme has been extensively studied by Banks and Burns [3] and Gibson [25]. The treatment of a more general case which includes multiple discrete delays and a continuously delayed term may be found in [3].

For any positive integer N, partition $[-r,0]$ into subintervals $[t_j^N, t_{j-1}^N]$ for $j = 1,2,\cdots,N$, where $t_j^N = -\frac{jr}{N}$. Let χ_j^N denote the characteristic function on $[t_j^N, t_{j-1}^N)$ for $j = 2,3,\cdots,N$, and χ_1^N the characteristic function on $[-r/N,0]$. Define the finite dimensional subspaces $Z^N \subset Z$ by

$$Z^N = \{(\eta,\phi)\in Z \mid \eta \in \mathbb{R}^n,\ \phi = \sum_{j=1}^{N} v_j\, \chi_j^N,\ v_j \in \mathbb{R}^n\}$$

and the projections $P^N:Z \to Z^N$ by

$$P^N(\eta,\phi) = \left(\phi_0^N, \sum_{j=1}^{N} \phi_i^N \chi_j^N\right)$$

where

$$\phi_0^N = \eta,\ \phi_i^N = \frac{N}{r} \int_{t_j^N}^{t_{i-1}^N} \phi(s)ds, \quad \text{for } j = 1,2,\cdots,N.$$

Let the operators $A^N:Z \to Z^N$ be defined by

$$A^N(\eta,\phi) = \left(A_0\ \phi_0^N + A_1\ \phi_N^N, \frac{N}{r} \sum_{j=1}^{N} (\phi_{j-1}^N - \phi_j^N)\chi_j^N\right).$$

If $T^N(t)$ denotes the semigroup generated by A^N, then the approximation to the abstract formulation of (4.13) is

$$z^N(t) = T^N(t)P^N(\eta,\phi) + \int_0^t T^N(t - \sigma)(B_0\ u(\sigma),0)d\sigma.$$

Since A^N is bounded for each N, this is equivalent to the initial value problem

$$\begin{aligned} \dot{z}^N(t) &= A^N\ z^N(t) + (B_0\ u(t),0) \\ z^N(0) &= P^N(\eta,\phi) \end{aligned} \tag{4.14}$$

Note that the operator A^N is reduced by the Hilbert space Z^N for each N. Since $(B_0u(t),0)\epsilon Z^N$ for every N, (4.14) is a differential equation in a finite dimensional space. Upon choosing the appropriate basis, equation (4.14) has the representation (see [3])

$$\dot{w}^N(t) = A^N w^N(t) + B^N u(t) \tag{4.15}$$
$$w^N(0) = \mathrm{col}(\phi_0^N, \phi_1^N, \cdots, \phi_N^N) ,$$

where A^N is the $n(N + 1) \times n(N + 1)$ matrix

$$\begin{bmatrix} A_0 & 0 & \cdots & 0 & A_1 \\ \frac{N}{r} I & -\frac{N}{r} & & & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{N}{r} I & -\frac{N}{r} I \end{bmatrix} , \tag{4.16}$$

I is the $n \times n$ identity matrix, and B^N is the $n(N + 1) \times m$ matrix $B^N = \mathrm{col}(B_0, 0, \cdots, 0)$.

The Nth approximate control problem becomes: find the $[u^N]^* \in L_2(0, t_f; \mathbb{R}^M)$ which minimizes

$$J^N(w^N(0), u^N) = \int_0^{t_f} (\langle Q^N w^N(s), w^N(s)\rangle + \langle Ru^N(s), u^N(s)\rangle) ds, \tag{4.17}$$

where $w^N(t)$ is the solution to (4.15). The matrix Q^N is given by

$$Q^N = \begin{bmatrix} Q & 0 & \cdots & 0 \\ 0 & & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & & 0 \end{bmatrix}$$

where 0 denotes the appropriate $n \times n$ zero matrix.

In [3] the authors proved that the AVE scheme satisfies a Trotter-Kato type approximation theorem. This established the strong convergence of the semigroups $T^N(t)$ to $T(t)$ defined by (4.3). Recall that one of the requirements of $\mathcal{H}_1$ is that the adjoint semigroups $[T^N(t)]^*$ converge strongly to $T^*(t)$. This result was obtained by Gibson [25] and is the key fact used to prove Lemma 3.2. The special forms of B^N and Q^N immediately satisfy $\mathcal{H}_0, \mathcal{H}_1$, and $\mathcal{H}_2$; thus the solutions to the Chandrasekhar equations (3.14) - (3.15) of the AVE approximation converge to the solutions of the Chandrasekhar integral equations (3.16) - (3.17) in the strong sense.

Before demonstrating that approximation of the gain operator via the Chandrasekhar equations has significant computational advantages over approximation of the gain operator via the Riccati equation, we introduce an additional reduction technique which takes advantage of a special structure that frequently occurs in hereditary systems. For these special systems, this technique will further reduce the number of differential equations that it is necessary to solve in order to approximate the gain operator, $K(t)$.

In applications of hereditary control problems, the delay does not always appear in each component of the state (see Examples 5.1, 5.2, and 5.3). In this type of system the initial data actually lie in a "smaller" state space. The AVE scheme as described projects each of the n components of the history portion of $(x(t),x_t)$ into an N-dimensional space. If only $q < n$ of the components are delayed, it should be possible to project the history portion into a space of dimension $q \times N$ instead of a space of dimension $n \times N$. The following discussion develops this idea.

The reduction as discussed here was originally studied in Cliff and Burns [17] for spline approximations in the context of parameter identification, and then for AVE in [10]. This reduction is a special case of the F-reduction discussed in [21] and [22].

For the remainder of this section it will be assumed that the matrix A_1 defined in equation (4.13) has the form

$$A_1 = \begin{bmatrix} 0 & A_{11} \\ 0 & A_{12} \end{bmatrix},$$

where A_{11} is $p \times q$, A_{12} is $q \times q$, and $p + q = n$. We shall write $x(t) = \mathrm{col}(w(t), y(t)) \in \mathbb{R}^p \times \mathbb{R}^q$ and not distinguish between columns and rows for ease of notation. Also, we shall not distinguish between the space Z defined earlier in this section and $\mathbb{R}^p \times \mathbb{R}^q \times L_2(-r,0;\mathbb{R}^p) \times L_2(-r,0;\mathbb{R}^q)$.

Let Z_R denote the "reduced space" $\mathbb{R}^p \times \mathbb{R}^q \times L_2(-r,0;\mathbb{R}^q)$. Define the projection operator $\mathcal{P}: Z \to Z_R$ by

$$\mathcal{P}(\eta_1,\eta_2,\phi_1,\phi_2) = (\eta_1,\eta_2,\phi_2),$$

and the injection $\mathcal{I}: Z_R \to Z$ by

$$\mathcal{I}(\eta_1,\eta_2,\phi_2) = (\eta_1,\eta_2,\hat{\eta}_1,\phi_2),$$

where $\hat{\eta}_1$ is the function in $L_2(-r,0;\mathbb{R}^p)$ with constant value η_1. For $\mathcal{A}$ and $T(t)$ as defined by (4.3) - (4.5) define the reduced operator $\mathcal{A}_R: \mathcal{D}(\mathcal{A}_R) \subset Z_R \to Z_R$ by

$$\mathcal{A}_R = \mathcal{P}\mathcal{A}\mathcal{I}, \tag{4.18}$$

with domain

$$\mathcal{D}(A_R) = \{(\eta_1,\eta_2,\phi_2)\in Z_R \mid \phi_2\in w^{1,2}, \qquad \eta_2 = \phi_2(0)\}. \tag{4.19}$$

In [10], Burns and Cliff stated but did not prove the following theorem. We include the proof for completeness.

THEOREM 4.2: *The operator* A_R *is the generator of a* C_0*-semigroup* $\{T_R(t)\}_{t\geq 0}$ *on* Z_R, *and* $T_R(t) = \mathcal{P}T(t)\mathcal{I}$.

Proof: Let $\mathcal{I}_R$ be the identity operator on Z_R. It is easy to show that $\mathcal{P}$ and $\mathcal{I}$ are bounded and it follows that $\mathcal{P}T(t)\mathcal{I}$ is strongly continuous in t. Also, since $\mathcal{P}$ is a left inverse for $\mathcal{I}$, we have that $\mathcal{P}T(0)\mathcal{I} = \mathcal{I}_R$. In order to show that $\mathcal{P}T(t)\mathcal{I}$ satisfies the semigroup property, first note that the solution to (4.13) (with $u = 0$) is independent of the initial choice ϕ_1. Hence, for any ϕ_1, $\overline{\phi}_1\in L_2(-r,0;\mathbb{R}^p)$, the equality

$$T(t)(\eta_1,\eta_2,\phi_1,\phi_2) = T(t)(\eta_1,\eta_2,\overline{\phi}_1,\phi_2), \qquad t \geq 0 \tag{4.20}$$

is obtained. Let $(w(t),y(t),w_t,y_t)$ denote the solution to (4.13) corresponding to $(\eta_1,\eta_2,\hat{\eta}_1,\phi_2)$ and $u \equiv 0$ (recall with our notation $x(t) = \mathrm{col}(w(t),y(t))$). For t_1 and $t_2 \geq 0$,

$$\begin{aligned} \mathcal{P}T(t_1)\mathcal{I}\mathcal{P}T(t_2)\mathcal{I}(\eta_1,\eta_2,\phi_2) &= \mathcal{P}T(t_1)\mathcal{I}\mathcal{P}T(t_2)(\eta_1,\eta_2,\hat{\eta}_1,\phi_2) \\ &= \mathcal{P}T(t_1)\mathcal{I}\left(w(t_2),y(t_2),y_{t_2}\right) \\ &= \mathcal{P}T(t_1)\left(w(t_2),y(t_2),\hat{w}(t_2),y_{t_2}\right). \end{aligned} \tag{4.21}$$

In view of (4.20), (4.21) becomes

$$\begin{aligned}\mathcal{P}T(t_1)\mathcal{I}\mathcal{P}T(t_2)\mathcal{I}(\eta_1,\eta_2,\phi_2) &= \mathcal{P}T(t_1)\big(w(t_2),y(t_2),w_{t_2},y_{t_2}\big)\\ &= \mathcal{P}\big(w(t_1+t_2),y(t_1+t_2),w_{t_1+t_2},y_{t_1+t_2}\big)\\ &= \big(w(t_1+t_2),y(t_1+t_2),y_{t_1+t_2}\big).\end{aligned}$$

A straightforward calculation yields

$$\mathcal{P}T(t_1+t_2)\mathcal{I}(\eta_1,\eta_2,\phi_2) = \big(w(t_1+t_2),y(t_1+t_2),y_{t_1+t_2}\big),$$

and the semigroup property is satisfied. Using standard agruments one can show that the domain of the infinitesimal generator of $\mathcal{P}T(t)\mathcal{I}$ is given by (4.19). For each $(\eta_1,\phi_2(0),\phi_2)\epsilon\mathcal{D}(\mathcal{A}_R)$ it follows that

$$\lim_{t\to\infty^+}\frac{\mathcal{P}T(t)\mathcal{I}-\mathcal{I}_R}{t}(\eta_1,\phi_2(0),\phi_2) = \big(A_0(w(0),y(0)) + A_1(w(-r),y(-r)),\dot{y}_t\big).$$

It is easily verified that

$$\mathcal{A}_R(\eta_1,\phi_2(0),\phi_2) = \big(A_0(w(0),y(0)) + A_1(w(-r),y(-r)),\dot{y}_t\big)$$

for $(\eta_1,\phi_2(0),\phi_2)\epsilon\mathcal{D}(\mathcal{A}_R)$; therefore $\mathcal{A}_R$ is the infinitesimal generator of $\mathcal{P}T(t)\mathcal{I}$ and $T_R(t) = \mathcal{P}T(t)\mathcal{I}$.

The "reduced" abstract evolution equation becomes

$$z_R(t) = T_R(t)(\eta_1,\eta_2,\phi_2) + \int_0^t T_R(t-s)\mathcal{B}_R\, u(s)ds \tag{4.22}$$

where $\mathcal{B}_R\, u = (B_0\, u, 0)$ (here $B_0\, u \in \mathbb{R}^n$ should be written as an element of $\mathbb{R}^p \times \mathbb{R}^q$). Theorem 4.2 and the action of the operators $\mathcal{P}$ and $\mathcal{I}$ yield the equivalence of (4.6) to (4.22), and hence $z_R(t) = (w(t),y(t),y_t)$ with $x(t) = \mathrm{col}(w(t),y(t))$. The cost functional for the control problem becomes (4.7) with the zero operators in $\tilde{Q}$ adjusted to the appropriate spaces. As a result of these observations, we conclude that solving the optimal control problem associated with (4.20) is equivalent to solving the optimal control problem of the original system (4.14).

Define the reduced AVE operators $\mathcal{A}_R^N$ by $\mathcal{A}_R^N = \mathcal{P}\mathcal{A}^N\mathcal{I}$. Computations similar to those in the proof of Theorem 4.2 reveal that the semigroup $T_R^N(t)$ generated by $\mathcal{A}_R^N$ satisfies $T_R^N(t) = \mathcal{P}T^N(t)\mathcal{I}$ and furthermore, it is easily verified that $\mathcal{A}_R^N$ and $T_R^N(t)$ are the same operators that result if the AVE scheme is applied directly to (4.21). For $(\eta_1,\eta_2,\phi_2) \in Z_R$, it follows that

$$\begin{aligned} T_R^N(t)(\eta_1,\eta_2,\phi_2) &= \mathcal{P}T^N(t)\mathcal{I}(\eta_1,\eta_2,\phi_2) \\ &= \mathcal{P}(T^N(t)(\eta_1,\eta_2,\hat{\eta}_1,\phi_2)). \end{aligned}$$

As commented earlier, Banks and Burns [3] proved that $T^N(t)z \to T(t)z$ uniformly in t on compact intervals for each $z \in Z$; thus $T^N(t)(\eta_1,\eta_2,\hat{\eta}_1,\phi_2) \to T(t)(\eta_1,\eta_2,\hat{\eta}_1,\phi_2)$. Since $\mathcal{P}$ is bounded, $T_R^N(t) \to T_R(t)$ strongly, and uniformly in t on compact intervals. These results are summarized in the following theorem.

THEOREM 4.3: *Let* $z_R^N(t)$ *denote the solution of the reduced AVE scheme approximation to (4.22) and suppose that* $z_R(t) = (w(t), y(t), y_t)$ *where* $x(t) = \mathrm{col}(w(t), y(t))$ *is the solution to (4.1) with initial data* (η_1, η_2, ϕ_2). *Then* $z_R^N(t) \to z_R(t)$ *in* Z_R, *and the convergence is uniform in* t *on compact intervals*.

If the AVE scheme is applied to the reduced equation (4.20) and the appropriate basis chosen, then the system of equations

$$[\dot{w}_R^N(t)] = A_R^N \, w_R^N(t) + B_R^N \, u(t)$$

$$w_R^N(0) = \mathrm{col}\big(\eta_1, \eta_2, (\phi_2)_1^N, \cdots, (\phi_2)_N^N\big)$$

is obtained. The $(n + qN) \times (n + qN)$ matrix A_R^N has the form

$$A_R^N = \left[\begin{array}{cc|cccc} & & & & & A_{11} \\ & A_0 & 0 & \cdots & \cdots & 0 \\ & & & & & A_{12} \\ \hline \Theta & \frac{N}{r} I & -\frac{N}{r} I & & & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & & \frac{N}{r} I & -\frac{N}{r} I \end{array}\right]$$

where I is the $q \times q$ identity matrix, Θ is the $q \times p$ zero matrix, and 0 represents the zero matrix of appropriate size. The reduced scheme has effectively discarded the zero columns of A_1 that appear in (4.16).

The strength of the Chandrasekhar algorithm lies in a comparison between the number of differential equations in its approximation to the number of differential equations in the approximation of the Riccati operator. We now demonstrate that the Chandrasekhar algorithm drastically reduces the number of equations that it is necessary to solve in order to approximate the gain operator. Moreover, if the structure of the hereditary system is such that the additional reduction technique discussed above applies, then further computational reductions occur.

Let the rank of the matrix Q in (4.2) be p_0. For the AVE scheme, the approximations Q^N also have rank p_0 and it is possible to obtain a factorization $Q^N = [V^N]^T V^N$ where V^N is a $p_0 \times n(N+1)$ matrix. Thus the set of Chandrasekhar equations associated with (4.15) and (4.17) contains

$$C(N) \equiv (m + p_0)[n(N+1)]$$

equations. The Riccati differential equation for (4.15), (4.17) has the form (see Gibson [25])

$$\begin{aligned}[\dot{P}^N(t)] &= -Q^N - [A^N]^T P^N(t) - P^N(t)A^N \\ &\quad + P^N(t)B^N[R]^{-1}[B^N]^T P^N(t) \qquad (4.23) \\ P^N(t_f) &= 0.\end{aligned}$$

Taking into account the symmetry of the the system, (4.23) contains

$$R(N) \equiv \frac{n(N+1)[n(N+1)+1]}{2}$$

differential equations. Similar counts for the reduced system yields

$$C_R(N) = (m + p_0)(n + qN)$$

and

$$R_R(N) = \frac{(n + qN)(n + qN + 1)}{2} ,$$

respectively.

The special form of Q^N yields $p_0 \leq n$ and thus p_0 is independent of N. Tables (4.1) and (4.2) give a comparison of the sizes of the different systems for various values of the parameters. The parameters in Table (4.2) are for the wind tunnel problem of Example 5.2.

Table 4.1: $n = 2,\ q = 1,\ m = 1,\ p_0 = 2.$

N	R(N)	C(N)	$R_R(N)$	$C_R(N)$
16	595	102	171	54
32	2775	198	595	102
64	8515	390	2211	198

Table 4.2: $n = 3$, $q = 1$, $m = 1$, $p_0 = 1$.

N	R(N)	C(N)	$R_R(N)$	$C_R(N)$
16	1326	102	190	38
32	4550	198	630	70
64	19,110	390	2278	134

Based on an equations count only, for sufficiently large N (which is small for most problems) the Chandrasekhar equations should offer substantial savings in time necessary to compute the gain. A comparison of R(N) to C(N) shows a reduction in the number of equations from $\mathcal{O}(N^2)$ to $\mathcal{O}(N)$. If in addition the F-reduction technique applies, then the number of equations is further reduced by a factor approximately equal to q/n.

A few remarks concerning generalizations of the problem discussed above are worthwhile. The inclusion of multiple discrete delays in equation (4.13) does not reduce the effectiveness of the Chandrasekhar equations. These additions affect the size and form of A^N given by (4.16) (see [3]). In particular, for 2 delays A^N becomes an $n[2N + 1] \times n[2N + 1]$ matrix, and the equation counts above are correct if C(N) is replaced by C(2N), R(N) by R(2N), etc. The presence of a continuous delay term does not affect these equation counts; it only affects the form of A^N. A numerical example involving two delays is given in Section V.

Another generalization would be to include a term of the form $\langle Gx(t_f), x(t_f)\rangle$ in the cost functional (4.2), which penalizes the final state. The addition of this term is motivated by the fact that in the finite dimensional case the inclusion of such a term has virtually no effect on the derivation of the Chandrasekhar equations other than to change the final values of $K(t)$ and $L(t)$ (see [14], [31]). In this case, the Chandrasekhar equations for the AVE approximation scheme may be written

$$\dot{K}^N(t) = R^{-1}[B^N]^T[L^N(t)]^T\, CL^N(t)$$

$$K^N(t_f) = R^{-1}[B^N]^T\, G^N$$

$$\dot{L}^N(t) = -L^N(t)[A^N - B^N\, K^N(t)]$$

$$L^N(t_f) = B_1^N$$

where

$$G^N = \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix},$$

0 being the appropriate zero matrix and B_1^N and C are matrices satisfying

$$[B_1^N]^T\, CB_1^N = -Q^N - [A^N]^T\, G^N - G^N\, A^N + G^N\, B^N\, R^{-1}[B^N]^T\, G^N.$$

We point out that in the case where only discrete delays appear, the special structures of A^N, B^N, G^N, and Q^N allow us to find C independent of N. In this case the matrices B_1^N have a constant rank p_0 with $p_0 \leq (k+1)n$ where k is the number of discrete delays. In the case of a continuous delay, it follows that for each N, $p_0 = p_0(N)$ and $C = C^N$ are functions of N. However, it still follows that $p_0(N) \leq (k+1)n$ for all N.

No claim is made concerning the convergence of these equations to an infinite dimensional form, but we do point out that the equations appear to converge numerically (see Examples 5.1 and 5.5), and it is interesting to note that an infinite dimensional version for this case has been derived in [27].

The use of the Chandrasekhar equations is not restricted to the AVE scheme. They may be applied to other schemes as well. Approximations using splines have become increasingly popular ([4], [6], and [7]). The spline scheme introduced by Banks and Kappel [6] does not satisfy the the strong convergence criteria on the adjoint semigroups that our work requires. However, a modification of this scheme by Kappel and Salamon [30] does. In fact, Kappel and Salamon have now applied a Chandrasekhar algorithm to their numerical examples and obtained satisfactory results. Another scheme for which the Chandrasekhar equations are applicable is the Legendre-tau approximation scheme introduced by Ito and Teglas in [28] and later applied to hereditary control systems in [29]. This scheme satisfies the strong convergence criteria on the adjoint semigroups and retains the low rank condition on Q^N which makes the implementation of the Chandrasekhar equations effective.

V. NUMERICAL EXAMPLES

In this section we present several examples which illustrate the numerical efficiency of the Chandrasekhar equations and the F-reduction technique when applied to delay-differential systems. These examples are presented to show the efficacy of the methods and are not intended to be a complete numerical test. The "applications" in this section are twofold. First, the two computational reduction techniques are applied to the optimal control problem on a finite interval. Secondly, since the infinite interval problem is the "limiting solution" of the finite interval problem under appropriate conditions, the Chandrasekhar approximations are integrated backward in order to obtain a "steady-state" value. This steady-state value is then used in the forward integration of the state equations, and the optimal control is thus approximated.

Each of the examples presented here was numerically solved on several different computers. In order to make a consistent comparison between examples, all of the computer run times recorded in this section will refer to the CPU time required on an IBM 3081 (located at VPI & SU). However, the plotted data given here may be from different machines, specifically, either a VAX 11/750 or the Cyber VPS-32 (Cyber 205 with enhanced memory located at NASA Langley Research Center) running in scalar mode. A standard fourth-order Runge-Kutta scheme was implemented to solve all differential equations, and unless stated otherwise, a fixed stepsize of $h = .01$ was chosen. For the finite interval optimal control problems, the Chandrasekhar equations were integrated backward from t_f and the gain values stored in increments of 10*h. In the forward integration of the state equations, linear interpolation was used to obtain the intermediate gain values.

Example 5.1: This problem has been numerically solved in [3] and [5] and an analytical solution given in [3]. The optimal control problem is to minimize

$$J(u) = 10x_2^2(2) + \int_0^2 u^2(s)ds$$

where

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t-1) \\ x_2(t-1) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

$$\begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} \equiv \begin{bmatrix} 10 \\ 0 \end{bmatrix} \quad \text{for} \quad -1 \leq s \leq 0.$$

The equation is the vector formulation of a harmonic oscillator with delayed damping given by

$$\ddot{y}(t) + \dot{y}(t-1) + y(t) = u(t).$$

The F-reduction technique combined with the Chandrasekhar algorithm was used to compute the optimal controls for $N = 8$, 16, 48, and 100. The total CPU times required for these computations are given in Table 5.1. Of particular note are the results for $N = 48$.

Table 5.1

N	4	8	16	48	100
CPU(sec)	.62	.94	1.57	4.18	8.3

In [3], the authors state that the CPU time required to approximate the optimal control by solving the Riccati equations for the AVE scheme with $N = 48$ was 3700 seconds. These computations were performed on an IBM 370/158 system at VPI & SU in 1974. The IBM 3081 on which the present results were obtained is approximately 6 - 7 times "faster" than the IBM 370/158 (this is a rough estimate given by the computer center at VPI & SU). The CPU time in [3] also included an additional computation to obtain J^{48} which the present results do not. The total CPU time for our computations was only 4.18 seconds, and even after taking into account the differences between the two computers, the reductions are still quite substantial. The reduction in CPU time may be credited to the reduction in the number of differential equations solved. In [3], the authors obtained the gain via a Riccati equation and thus had to solve 4851 differential equations in addition to 98 state equations. Using the techniques discussed in this paper, it is necessary to solve only 150 differential equations to obtain the gain and 50 additional equations to obtain the optimal trajectories.

Figures 5.1 - 5.3 illustrate the convergence of the approximation $[u^N(t)]^*$ to the optimal control $u^*(t)$. Note that in Figure 5.3 the approximation $[u^{100}(t)]^*$ follows the true solution closely until a time value of approximately 1. and then appears to lose accuracy. Implementing a smaller stepsize in the Runge-Kutta scheme corrected this.

Also included here are graphs of the approximations to the functional gain $a \to K_1(t,a)$, of the gain operator $K(t)$. Recall that the averaging scheme produces piecewise constant approximations to functions in $L_2(-r,0)$; thus the elements ◇ and Δ in Figures 5.4 - 5.11 represent these constant values on the appropriate subintervals $[a_{i-1},a_i]$ of $[-r,0]$. In order to obtain illustrative clarity, these points were connected to form a smooth curve for $N = 100$. Pictured are approximations of the functional gains $K_1^N(t,a)$, for $N = 8, 32, 100$, at the values of time $t = 0, .25, .5, .75, 1.,$ $1.25, 1.5,$ and 1.75. These figures show not only the convergence in N of the functional gains, but also the evolution in time of these approximations. Note that the approximate functional gains calculated at $t = 1.25, 1.5,$ and

1.75 do not clearly indicate the discontinuities that actually appear in the functional gain due to the presence of a terminal cost term in the functional to be minimized (see Delfour [20]). This phenomenon will be discussed in more detail in Example 5.5.

Finally, graphs of the resulting state approximations for $N = 8$ and $N = 100$ which show rapid convergence of the states are given in Figures 5.12 and 5.13.

Example 5.2: This next example is an application of the Chandrasekhar equations and of the F-reduction technique to an infinite time optimal control problem. This illustrates that the Chandrasekhar equations may be a viable method of computing the constant gain associated with the infinite time problem. The motivation for studying this particular example is not only that the problem has the special structure which fits our framework, but that it is a practical problem as well. The system is a model for fine tuning the mach number in a cryogenic windtunnel (National Transonic Facility) constructed by NASA at its Langley Research Center in Hampton, Virginia. This model has been studied in [37], [39], and the LQR problem numerically approximated in [7], [33].

The controller of the system is an actuator attached to a wind guide vane and finely tunes the mach number by changing the angle of the vane. The state consists of the variation of the mach number, the variation in the guide vane angle velocity, and the variation of the guide vane angle. The equation has the form

$$\dot{x}(t) = \begin{bmatrix} -a & 0 & 0 \\ 0 & -2b\omega & -\omega^2 \\ 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 & ka \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t - .33) + \begin{bmatrix} 0 \\ \omega^2 \\ 0 \end{bmatrix} u(t),$$

$$x(s) = \mathrm{col}(-.1, 0., 8.547), \qquad -.33 \leq s \leq 0,$$

where the parameters $1/a$, ω^2, k, and b have the values 1.964, 36., -.0117, and .8, respectively. The quantity to be minimized is

$$J(u) = \int_0^\infty (x^T(s)Qx(s) + u^2(s))ds,$$

where $Q = \mathrm{diag}(10^4,0,0)$.

In [7], the authors were comparing the AVE approximation scheme to a spline approximation scheme. They applied a Newton iteration scheme to the Riccati matrix equations that resulted when the AVE scheme was used (for N = 2, 4, and 8). However, the Newton iteration scheme did not converge to the solution of the Riccati equation for $N = 8$.

In our approach, we solved the finite time optimal control problem on an interval $[0,t_f]$ of sufficient length so that the gain $K^N(t)$ satisfied $K^N(t) \to K_0^N$ as $t \to 0^+$, (i.e., we integrated backward to a steady-state solution K_0^N). This method resulted in a convergent scheme for $N = 8$, and in fact, for larger values of N as well.

Figure 5.14 illustrates the convergence of the functional gain $K_1^{48}(0,a)$ to a steady-state value for the values of t_f equal to 2, 3, 5, and 10. Note the rapid convergence of the gain to a steady-state value and that the values for $t_f = 5$ and 10 are almost indistinguishable.

The states and optimal control computed by using the steady-state gain, $K^{48}(0)$ are given in Figures 5.15 - 5.18. As can be seen, the response of the system is very good, and the states are driven to zero rapidly. These results agree well (pictorially) with those found in [7].

The total computer time required to compute the gain, optimal control, and states for this problem $(N = 48)$ was 7.76 seconds. This computation used both the Chandrasekhar equations and the additional reduction technique outlined earlier. The same problem was also solved on the computer using only the Chandrasekhar equations and not the additional reduction. In this case the total CPU time was 21.55 seconds, a significant increase.

Example 5.3: In this example the Chandrasekhar algorithm and F-reduction techniques are applied to the hereditary model of the two-dimensional airfoil discussed in Section 2. The resulting delay model has the form (see equations (2.16) and (2.17))

$$\dot{x}(t) = A_0\, x(t) + A_1\, x(t - s) + Bu(t)$$

(5.1)

$$x(s) = \phi(s) \qquad -r \leq s \leq 0$$

with output

(5.2) $$y(t) = Cx(t).$$

Here, equation (2.17) has been multiplied by the inverse of E. The matrices in (5.1) - (5.2) are

$$A_0 = \begin{bmatrix} -7.3865 & -33.8517 & -4479.50 & -5711.27 & 3.0804 \\ .9378 & -12.4685 & 568.73 & -7068.77 & -.3911 \\ 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ -2.7443 & 1404.29 & -1664.26 & -40701.67 & -253.856 \end{bmatrix}$$

$$A_1(5,5) = -47.00, \quad A_1(i,j) = 0, \quad (i,j) \neq (5,5)$$

$$B = [-82.333 \quad 191.286 \quad 0.0 \quad 0.0 \quad 864.533]^T$$

$$C = \mathrm{diag}[\sqrt{50.00} \quad \sqrt{50.0} \quad \sqrt{10.0} \quad \sqrt{10.0} \quad 1]$$

$$\phi(s) = \mathrm{col}[-.80 \quad .50 \quad .055 \quad .029 \quad 50.0], \quad s \in [-r,0],$$

and the time delay is $r = .05$. The cost functional that we desire to minimize is

$$J(u) = \int_0^{t_f} (y^T(t)y(t) + Ru^2(t))dt$$

where the final time is $t_f = .25$ sec., and $R = 10$.

Because of the small delay, the stepsize h used in the Runge-Kutta scheme was reduced to .001. The computations for this particular example exhibited in Figures 5.19 - 5.24 were performed on a VAX 11/750 computer using single precision arithmetic.

The parameters used to construct the above matrices are a slight variation of those found in [10]. In particular, the non-dimensionalized distance, x_α, from the airfoil center of mass to the elastic axis (e.a.) is .5 which places the e.a. at the quarter-chord point. In addition, the free-stream velocity U was chosen to be 1500 in/sec. This particular value was selected since it produced unbounded oscillations in our open loop system (a few numerical tests determined a "flutter speed" of approximately 1375 in/sec for this model). This unstable plant models the airfoil operating under flutter conditions, i.e., the pitch and plunge modes couple in a manner that allows unstable oscillations to occur. Since it is desirable to eliminate flutter, the controller should be able to damp effectively the pitching and plunging motions.

The reduced AVE scheme with $N = 16$ was implemented in order to approximate (5.1) - (5.2). The time-varying gain values produced by the Chandrasekhar equations were used to obtain the optimal control, $u^{16}(t)$. Graphs plotting the closed loop responses (dotted graph) of the LQR design are compared with the responses of the open loop (solid graph) system in Figures 5.19 - 5.23. As can be seen in the figures, the pitching and plunging motions (x_3 and x_4 respectively) and their velocities (x_1 and x_2) are quickly driven to zero in the closed loop response, illustrating the effectiveness of an LQR approach for our model. The optimal control which produced these responses is given in Figure 5.24.

The computational advantages of the reduction techniques discussed in this paper are very apparent in this example. For $N = 16$, in order to solve for the gain via the Riccati equation, one must solve 3,655 differential equations (this includes taking advantage of the symmetry of the equation); moreover, there are 85 state equations to solve. In contrast, if the two techniques discussed here are employed, it is necessary to solve only 126 equations to obtain the gain and 21 equations to obtain the state. If N is further increased to 48, then the number of equations becomes 30,135 Riccati and 245 state equations compared to 318 Chandrasekhar and 53 state equations. A similar problem (i.e., same system with different aerodynamic parameters) was solved on an IBM 3081. For $N = 16$, the total CPU time required by the methods here was 8.47 seconds. When N was increased to 48, the CPU time increased to 20.13 seconds.

Example 5.4: This example is a variation on Example 5.3. The parameters x_α and U defined previously are set equal to .297 and 1325., respectively. This value of x_α moves the e.a. from the quarter chord point (note also that from the figures U is below the "flutter speed"). The resulting matrices for equations (5.1) - (5.2) may be found in [12]; all other values are as in Example 5.3. The purpose of this example is to compare the state response obtained by using the steady-state value of the gain to the state response obtained by using the time-varying gain. In applications, use of a constant gain is often preferable since the gain values do not have to be continually updated at each instant of time.

The Chandrasekhar equations were integrated on the interval $[0,.25]$, and the value of $K^N(t)$ at $t = 0$ was used as the steady-state value. Figures 5.25 - 5.29 compare the open loop response to the closed loop response on $[0,.5]$ obtained by integrating the state equations using the constant gain. The closed loop responses produced by the time-varying gain yielded results that appeared graphically the same as the responses produced by the constant gain. There were, however, slight numerical differences. Graphs of the absolute values of these differences for $\dot{h}(t)$ and $\alpha(t)$ are given in

Figures 5.30 and 5.31. These graphs are representative of the results for all of the states. Figure 5.32 is included to show how the kernel portion of the functional gain $K_1^{48}(t,\alpha)$ evolves in time. In [1], the authors treated a similar problem. However, their results showed high frequency oscillations near $t = .25$, which may be due to numerical instabilties.

Example 5.5: This last example is a one-dimensional equation with two delays considered in [20]. The optimal control problem is to minimize

$$J = 10x^2(2) + \int_0^2 u^2(s)ds$$

where $x(t)$ satisfies

$$\dot{x}(t) = x(t - .5) + x(t - 1.) + u(t)$$

$$x(s) \equiv 1, \qquad -1 \leq s \leq 0.$$

This problem illustrates the type of discontinuities that may occur in the kernel of the gain, and how they are affected by the approximation scheme.

Since the problem has delays of 1. and .5, $K_1(t,a)$ will have a fixed discontinuity at $a = -.5$ for each t. However, because of the presence of a final cost term, $K_1(t,a)$ will also have discontinuities at (see Delfour [20]) $a_1 = -1. + t_f - t$ and $a_2 = -.5 + t_f - t$ (for values of t which place a_1 or a_2 in $[-1,0]$). The approximations $K_1^{100}(t,a)$ at $t = 1.9$, 1.4, 1., and 0. are given in Figures 5.33 - 5.36. Note that Figure 5.33 should show discontinuities at $a = -.4$ and $a = -.9$, and Figure 5.34 should show a discontinuity at $a = -.4$. The smoothing that occurs is due to the averaging of values which the Runge-Kutta scheme uses to produce each new iterate, and is not a characteristic of the Chandrasekhar algorithm. Though not exhibited here, use of a lower order Euler scheme (which does not average values for the next iterate) shows definite jumps at the points of discontinuity, but sacrifices overall accuracy. The effect of the smoothing of the kernel is minimized, however, since the action of the kernel is through an integral term. Consequently, accurate values for the optimal trajectories and optimal control are still obtained.

APPENDIX

PROOF OF LEMMA 3.1

Let $\hat{R}^N_s(t)$ and $[\hat{\mathcal{B}}^N_s]^*(t)$ be the Nth approximation analogues of (3.5) and (3.7). Note that $\hat{R}^N_s(t)$ is self-adjoint and satisfies $\|\hat{R}^N_s(t)\| \geq M$ for each N, s, and t. It follows that $\|[\hat{R}^N_s]^{-1}(t)\| \leq \frac{1}{M}$ uniformly in N, s, and t. Also, it follows that $(T^N_s\cdot)(t)$, $(\mathcal{T}^N_s\cdot)(t)$, and $(\mathcal{F}^N_s\cdot)$ are uniformly bounded in N, s, and t. Combining these observations with the uniform bounds on $\mathcal{B}^N$ and Q^N, it follows that $\|\hat{\mathcal{B}}^N_s(t)\|$ is uniformly bounded in N, s, and t. Therefore, $\|[\hat{R}^N_s]^{-1}\ \hat{\mathcal{B}}^{N*}_s(t)\|$ is also uniformly bounded. Recalling that $[\hat{R}^N_s]^{-1}(t)$ is self-adjoint and that the norm of a bounded operator equals the norm of its adjoint, we have established the lemma.

PROOF OF LEMMA 3.2

We first observe that strong, pointwise convergence of $T^N(t)$ and $[T^N(t)]^*$ along with an application of the dominated convergence theorem implies that $(\mathcal{T}^N_s\cdot)(t)$, $(\mathcal{T}^{N*}_s\cdot)(t)$, and $(\mathcal{F}^N_s\cdot)$ converge strongly and pointwise to $(\mathcal{T}_s\cdot)(t)$, $(\mathcal{T}^*_s\cdot)(t)$, and $(\mathcal{F}_s\cdot)$, respectively. In addition, $(T^N_s\cdot)(t)$, $(T^{N*}_s\cdot)(t)$, and $(\mathcal{F}^{N*}_s\cdot)(t)$ converge strongly and pointwise to $(T_s\cdot)(t)$, $(T^*_s\cdot)(t)$, and $(\mathcal{F}^*_s\cdot)(t)$, respectively. Since $\mathcal{B}^N$, and Q^N converge strongly, it follows that $\hat{R}^N_s(t) \rightarrow \hat{R}_s(t)$ and $\hat{\mathcal{B}}^{N*}_s(t) \rightarrow \hat{\mathcal{B}}^*_s(t)$ strongly and pointwise for $0 \leq s \leq t \leq t_f$. Taking the adjoint of (3.6) and using similar reasoning, it can be shown that $\hat{\mathcal{B}}^N_s(t) \rightarrow \hat{\mathcal{B}}_s(t)$ strongly and pointwise for $0 \leq s \leq t \leq t_f$. The identity $\hat{R}^{N^{-1}}_s(t) - \hat{R}^{-1}_s(t) = \hat{R}^{N^{-1}}_s(\hat{R}_s - \hat{R}^N_s)\hat{R}^{-1}_s(t)$ now implies that $\hat{R}^{N^{-1}}_s(t) \rightarrow \hat{R}^{-1}_s(t)$ strongly and pointwise. Therefore, it now follows that $\hat{\mathcal{B}}^N \hat{R}_s^{N^{-1}}(t) \to \hat{\mathcal{B}}\hat{R}_s^{-1}(t)$ strongly and pointwise for $0 \leq s \leq t \leq t_f$. As indicated previously, the convergence of $S^N(t,s)$ to $S(t,s)$ was shown by Gibson [25]. Using equation (3.8) we see that $S^{N*}(t,s)$ may be expressed as

$$(A.1) \qquad [S^N(t,s)]^* z = [T^N(t - s)]^* z - \int_s^t \hat{\mathcal{B}}_s [\hat{R}_s^N]^{-1}(\eta)[\mathcal{B}^N]^*[T^N(t - \eta)]^* z d\eta.$$

Combining the assumptions on $T^{N*}(t)$ and $\mathcal{B}^{N*}$ with Lemma 3.1, the integrand of (A.1) is seen to be uniformly bounded in N, s, and t. As a result of the comments above, the integrand also converges pointwise to $\hat{\mathcal{B}}_s \hat{R}_s^{-1}(\eta)\mathcal{B}^* T^*(z - \eta)z$, and an application of the dominated convergence theorem yields $S^{N*}(t,s) \to S^*(t,s)$ strongly and pointwise. The uniform boundedness of $S^{N*}(t,s)$ (and hence $S^N(t,s)$) follows directly from (A.1).

PROOF OF THEOREM 3.3

An integration of (3.14) results in the equation

$$(A.2) \qquad K^N(t)z = \int_t^{t_f} R^{-1}[\mathcal{B}^N]^*[L^N(\eta)]^* L^N(\eta)z d\eta.$$

Equation (3.15) may be rewritten as

$$\dot{L}^N(t)z = -L^N(t)A^N z + L^N(t)\mathcal{B}^N K^N(t)z$$

and a variation of parameters formula yields

$$L^N(t)z = V^N\, T^N(t_f - t)z - \int_t^{t_f} L(\eta)\mathcal{B}^N\, K^N(\eta)T^N(\eta - t)zd\eta. \tag{A.3}$$

Using the assumption that $L^N(t) = V^N\, S^N(t_f,t)$, (A.2) and (A.3) become

$$K^N(t)z = \int_t^{t_f} R^{-1}\, \mathcal{B}^{N*}\, S^{N*}(t_f,\eta)V^{N*}\, V^N\, S^N(t_f,\eta)zd\eta, \tag{A.4}$$

and

$$V^N\, S^N(t_f,t)z = V^N\, T^N(t_f - t)z - \int_t^{t_f} V^N\, S^N(t_f,\eta)\mathcal{B}^N\, K^N(\eta)T^N(\eta - t)zd\eta, \tag{A.5}$$

respectively. The dominated convergence theorem, Lemma 3.2, and Gibson's convergence results (see [25]) for $K^N(t)$ imply that the limit as $N \to \infty$ of (A.4) exists and

$$K(t)z = \int_t^{t_f} R^{-1}\, \mathcal{B}^*\, S^*(t_f,\eta)V^*\, VS(t_f,\eta)zd\eta. \tag{A.6}$$

Since $K^N(t)$ and $T^N(t)$ are each strongly continuous in t and converge to $K(t)$ and $T(t)$ in the strong sense, they are uniformly bounded in N and t on compact t-intervals. Therefore, another application of the dominated convergence theorem and Lemma 3.2 imply that the limit as $N \to \infty$ of (A.5) exists and

$$VS(t_f,t)z = VT(t_f - t)z - \int_t^{t_f} VS(t_f,\eta)\mathcal{B}K(\eta)T(\eta - t)zd\eta. \tag{A.7}$$

Defining $L(t) = VS(t_f,t)$ we have shown that the pair $K(t)$, $L(t)$ satisfy the Chandrasekhar equations (3.16) - (3.17) and $K^N(t)z \to K(t)z$ for all $z \in \mathcal{H}$. If $\mathcal{U}$ is finite dimensional, then it follows (see Gibson [25]) that $K^N(t) \to K(t)$ in the uniform operator topology.

PROOF OF THEOREM 3.4:

Let $(\hat{K}(t),\hat{L}(t))$ be another solution pair for (3.16) - (3.17). It follows from (3.16) that

$$\|(\hat{K}(t) - K(t))z\| \leq \int_t^{t_f} \|R^{-1}\mathcal{B}^*(\hat{L}^*(\eta)\hat{L}(\eta) - L^*(\eta)L(\eta))z\|d\eta$$

$$\leq \|R^{-1}\mathcal{B}^*\|\int_t^{t_f} \|\hat{L}^*(\eta)\hat{L}(\eta) - L^*(\eta)L(\eta)\|\ \|z\|d\eta,$$

which implies that

$$\|\hat{K}(t) - K(t)\| \leq \|R^{-1}\mathcal{B}^*\|\int_t^{t_f} \|\hat{L}^*(\eta)\hat{L}(\eta) - L^*(\eta)L(\eta)\|d\eta.$$

Adding and subtracting the appropriate term yields the inequality

$$\|\hat{K}(t) - K(t)\| \leq \|R^{-1}\mathcal{B}^*\|\int_t^{t_f} \big(\|\hat{L}^*(\eta)\hat{L}(\eta) - L^*(\eta)\hat{L}(\eta)\| + \|L^*(\eta)\hat{L}(\eta) - L^*(\eta)L(\eta)\|\big)d\eta$$

$$\leq \|R^{-1}\mathcal{B}^*\|\int_t^{t_f} \big(\|\hat{L}(\eta)\|\ \|\hat{L}^*(\eta) - L^*(\eta)\| + \|L^*(\eta)\|\ \|\hat{L}(\eta) - L(\eta)\|\big)d\eta.$$

Since the norm of an operator equals the norm of its adjoint, it follows that

$$\|\hat{K}(t) - K(t)\| \leq \|R^{-1}\mathcal{B}^*\|\int_t^{t_f} \big(\|\hat{L}(\eta)\| + \|L(\eta)\|\big)\ \|\hat{L}(\eta) - L(\eta)\|d\eta.$$

The principle of uniform boundedness implies the existence of a constant $M_1 \geq 0$ such that

(A.8) $$\|\hat{K}(t) - K(t)\| \leq M_1 \int_t^{t_f} \|L(\hat{\eta}) - L(\eta)\| d\eta.$$

Similarly, equation (3.17) yields the inequality

$$\begin{aligned}
\|\hat{L}(t) - L(t)\| &\leq \int_t^{t_f} \|\hat{L}(\eta)\mathcal{B}\hat{K}(\eta) - L(\eta)\mathcal{B}K(\eta)\| \; \|T(\eta - t)\| d\eta \\
&\leq \int_t^{t_f} \|T(\eta - t)\| \; \|\hat{L}(\eta)\mathcal{B}\hat{K}(\eta) - \hat{L}(\eta)\mathcal{B}K(\eta) + \hat{L}(\eta)\mathcal{B}K(\eta) \\
&\qquad\qquad - L(\eta)\mathcal{B}K(\eta)\| d\eta \\
&\leq \int_t^{t_f} \|T(\eta - t)\| \Big(\|\hat{L}(\eta)\mathcal{B}\| \; \|\hat{K}(\eta) - K(\eta)\| \\
&\qquad\qquad + \|\hat{L}(\eta) - L(\eta)\| \; \|\mathcal{B}K(\eta)\|\Big) d\eta.
\end{aligned}$$

Again, the principle of uniform boundedness can be applied to obtain the existence of $M_2 > 0$ such that

(A.9) $$\|\hat{L}(t) - L(t)\| \leq M_2 \int_t^{t_f} \big(\|\hat{L}(\eta) - L(\eta)\| + \|\hat{K}(\eta) - K(\eta)\|\big) d\eta.$$

Substituting (A.8) into (A.9) yields

(A.10) $$\|\hat{L}(t) - L(t)\| \leq M_2 \int_t^{t_f} \Big(\|\hat{L}(\eta) - L(\eta)\| + M_1 \int_\eta^t \|\hat{L}(s) - L(s)\| ds\Big) d\eta.$$

Since $t \leq \eta$ in (A.10), the lower limit may be extended to obtain

$$\|\hat{L}(t) - L(t)\| \leq M_2 \int_t^{t_f} \|\hat{L}(\eta) - L(\eta)\| d\eta + M_1 M_2 \int_t^{t_f} \int_t^{t_f} \|\hat{L}(s) - L(s)\| ds d\eta.$$

Applying Fubini's theorem and interchanging the order of integration, we have

$$\|\hat{L}(t) - L(t)\| \leq (M_2 + M_1 M_2(t_f - t)) \int_t^{t_f} \|\hat{L}(s) - L(s)\| ds,$$

and hence

$$\|\hat{L}(t) - L(t)\| \leq (M_2 + M_1 M_2 t_f) \int_t^{t_f} \|\hat{L}(s) - L(s)\| ds. \qquad \text{(A.11)}$$

In order to justify the existence of each of the integrals appearing above, we note that each operator is strongly continuous on $[0,t_f]$, and hence the norms of these operators are lower semi-continuous on $[0,t_f]$ (see Kato [34]). Lower semi-continuity implies measurability, and since the norms are uniformly bounded, they are integrable.

Gronwall's inequality holds for integrable functions almost everywhere (Reid [41]), hence (A.11) and Gronwall's inequality imply that $\|\hat{L}(t) - L(t)\| = 0$ almost everywhere in $[0,t_f]$. Since $\|\hat{L}(t)z - L(t)z\|$ continuous for each $z \in H$ and equal to zero almost everywhere, it follows that $\hat{L}(t) = L(t)$ everywhere on $[0,t_f]$. Equation (A.8) immediately yields $\hat{K}(t) = K(t)$ for $t \in [0,t_f]$, and the proof is complete.

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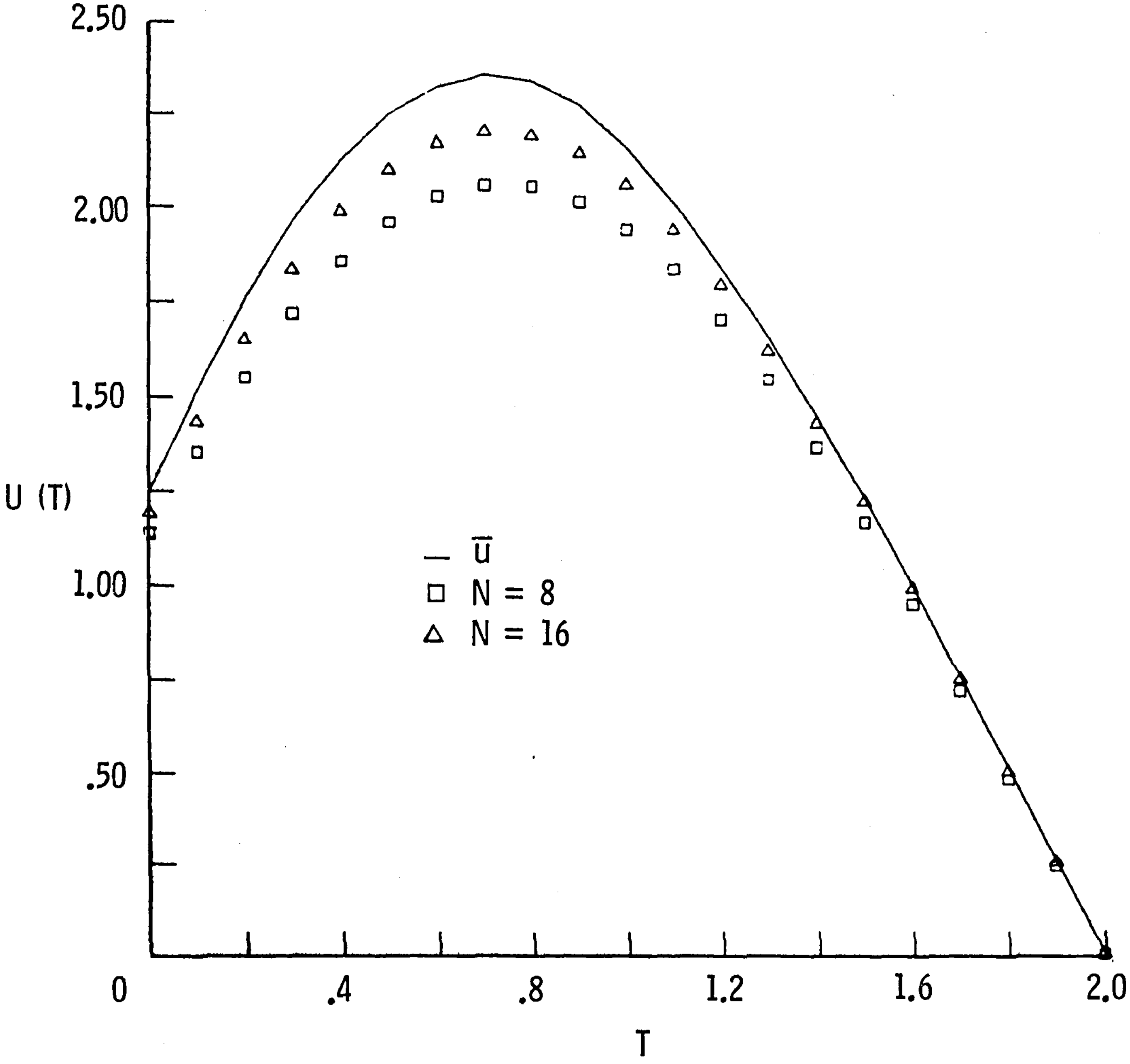

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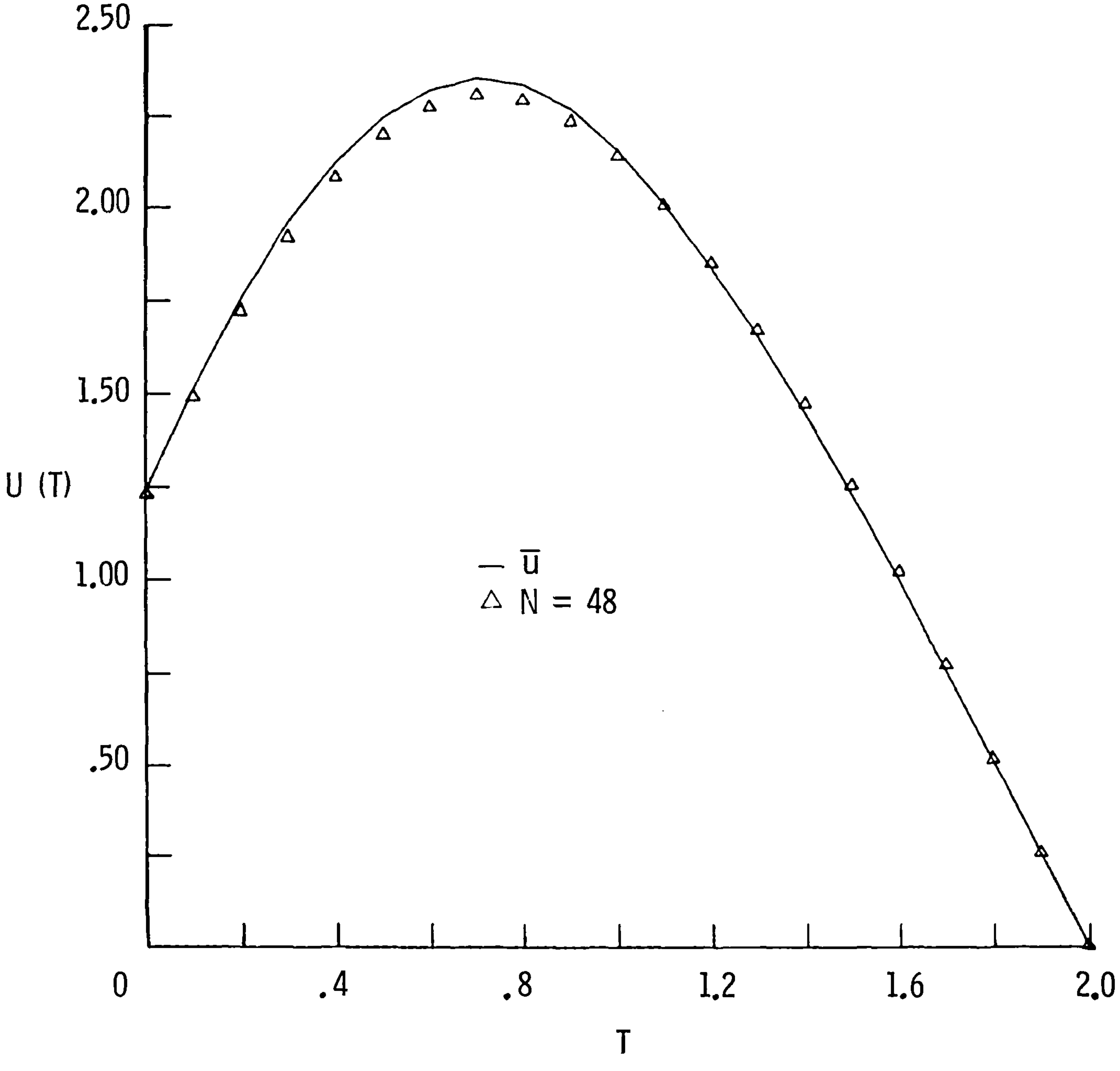

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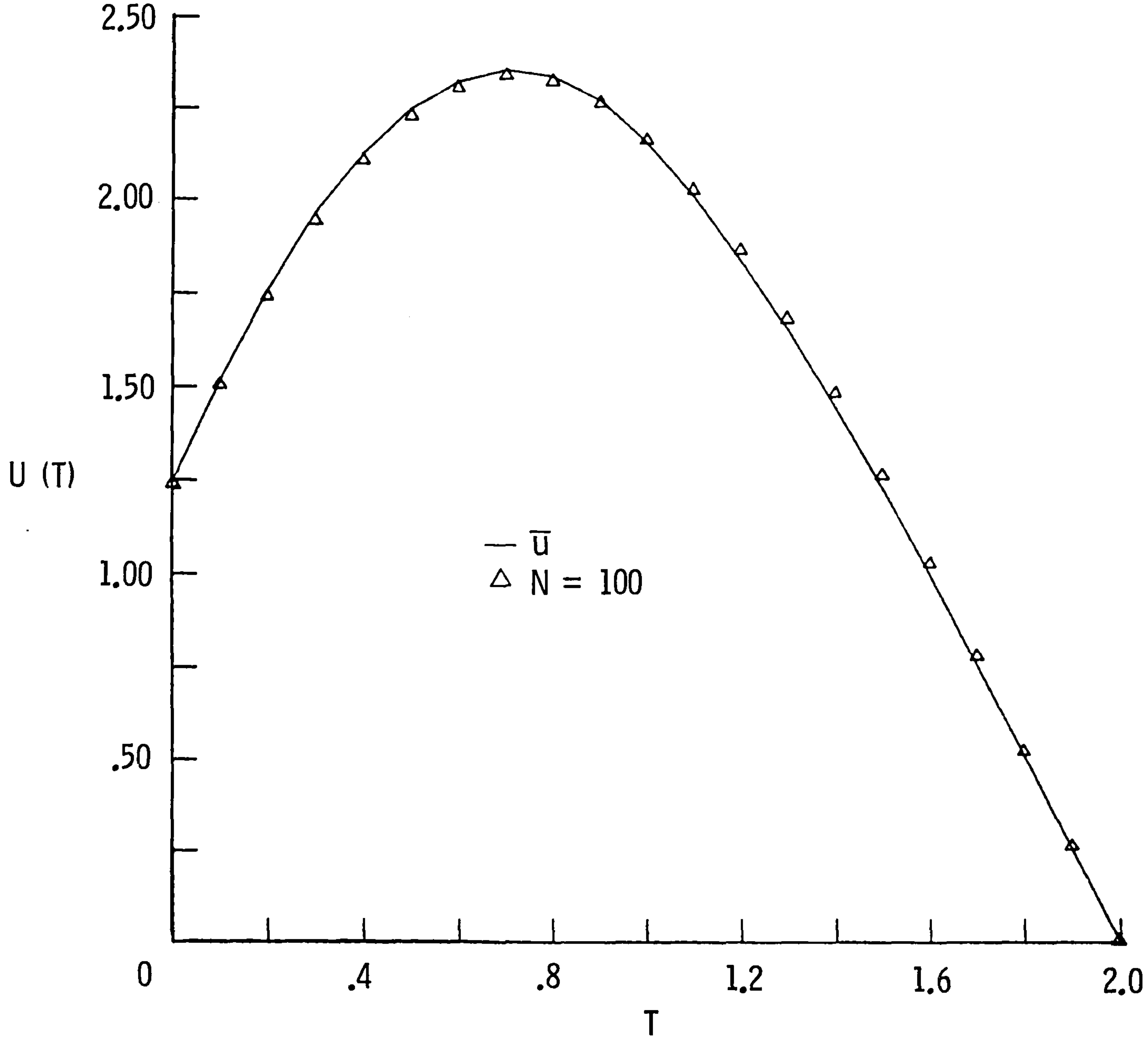

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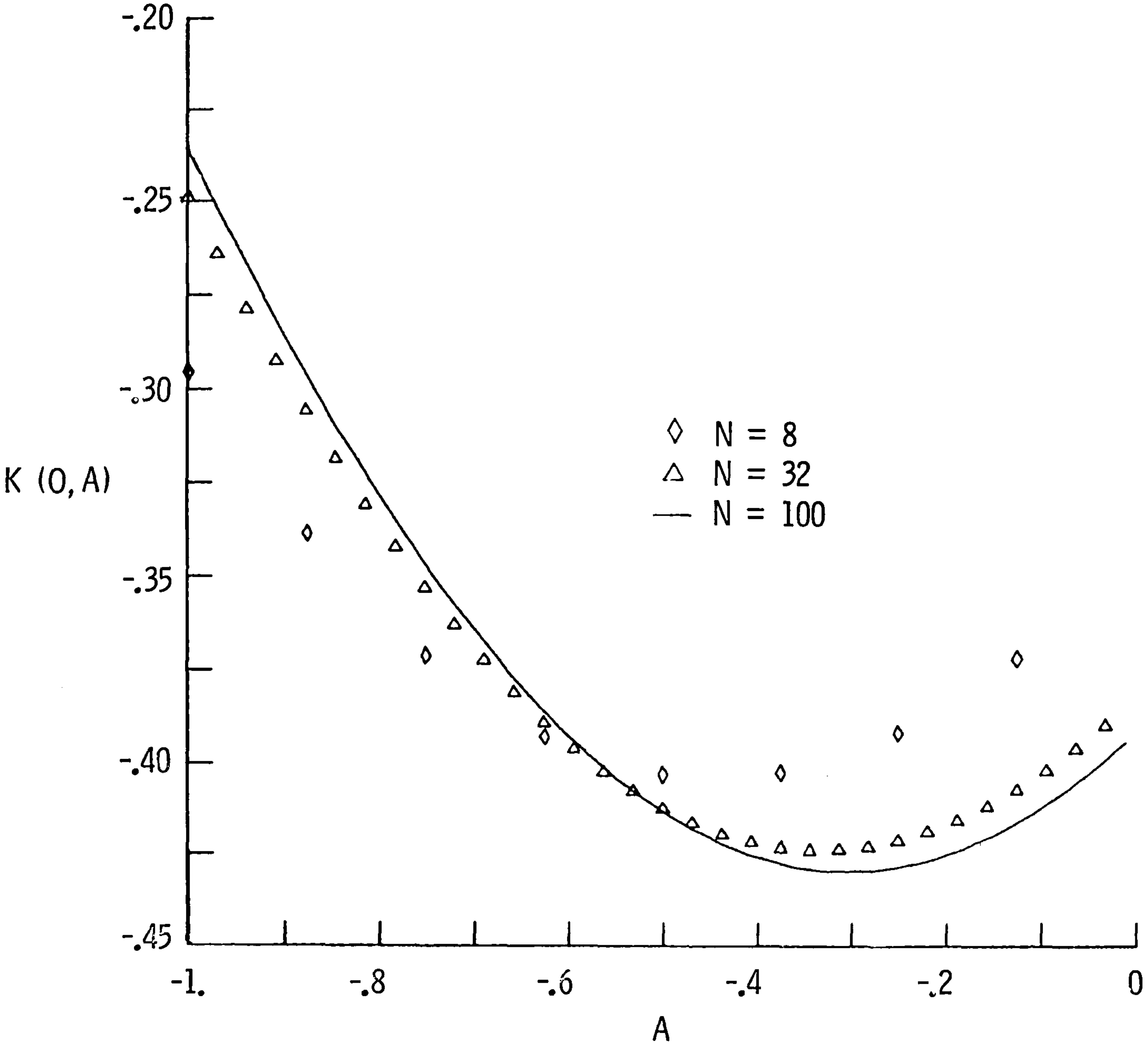

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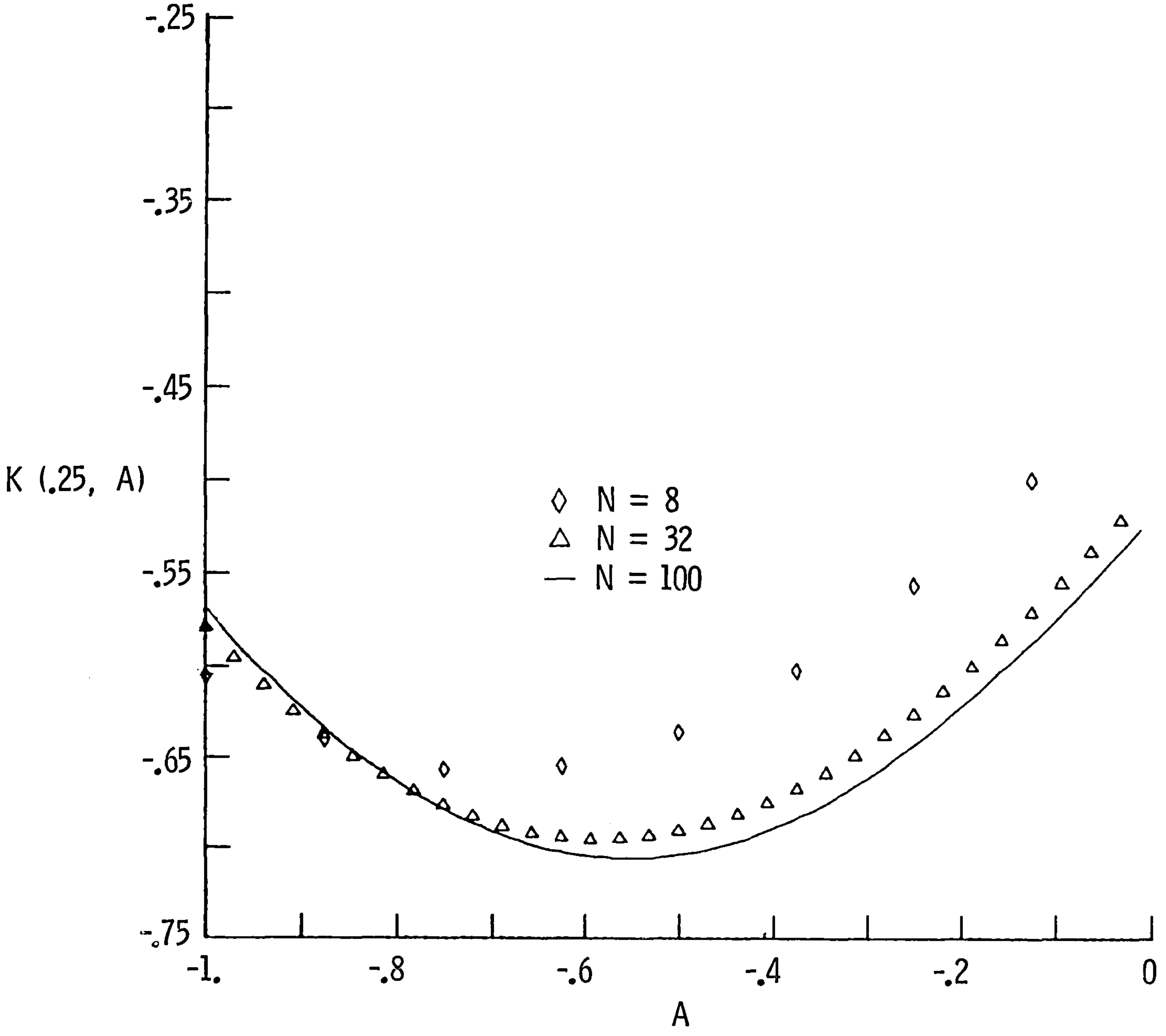

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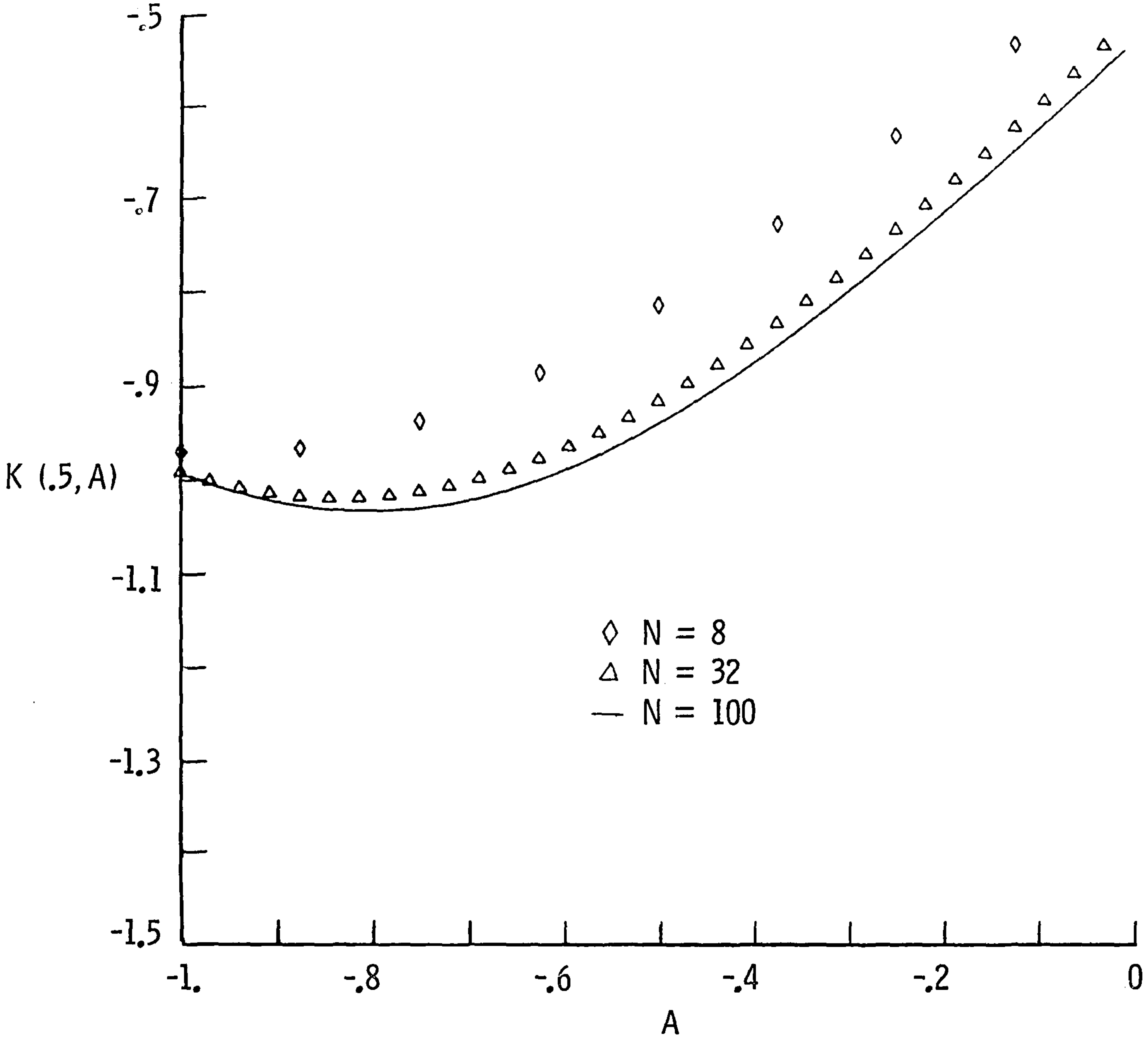

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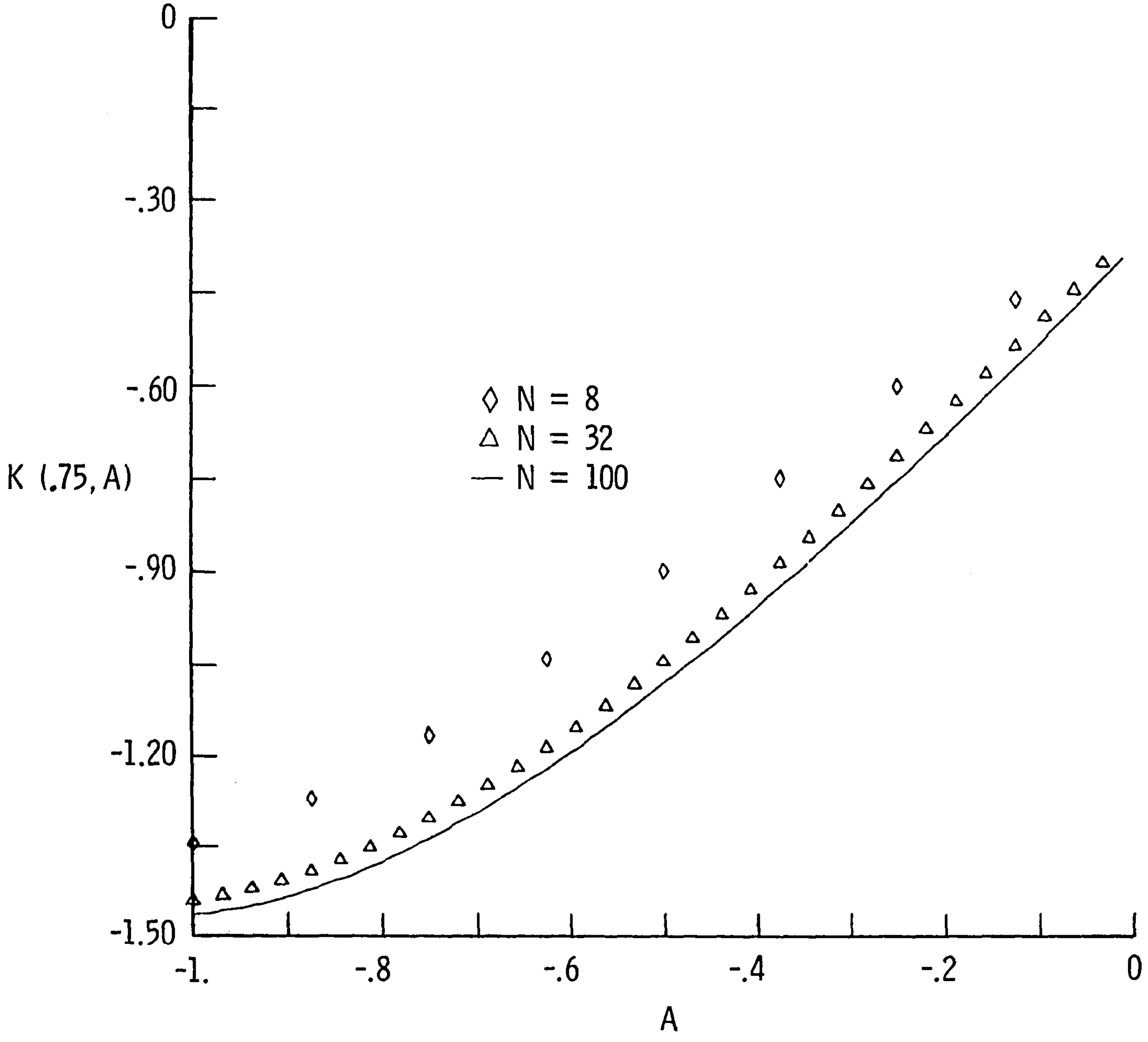

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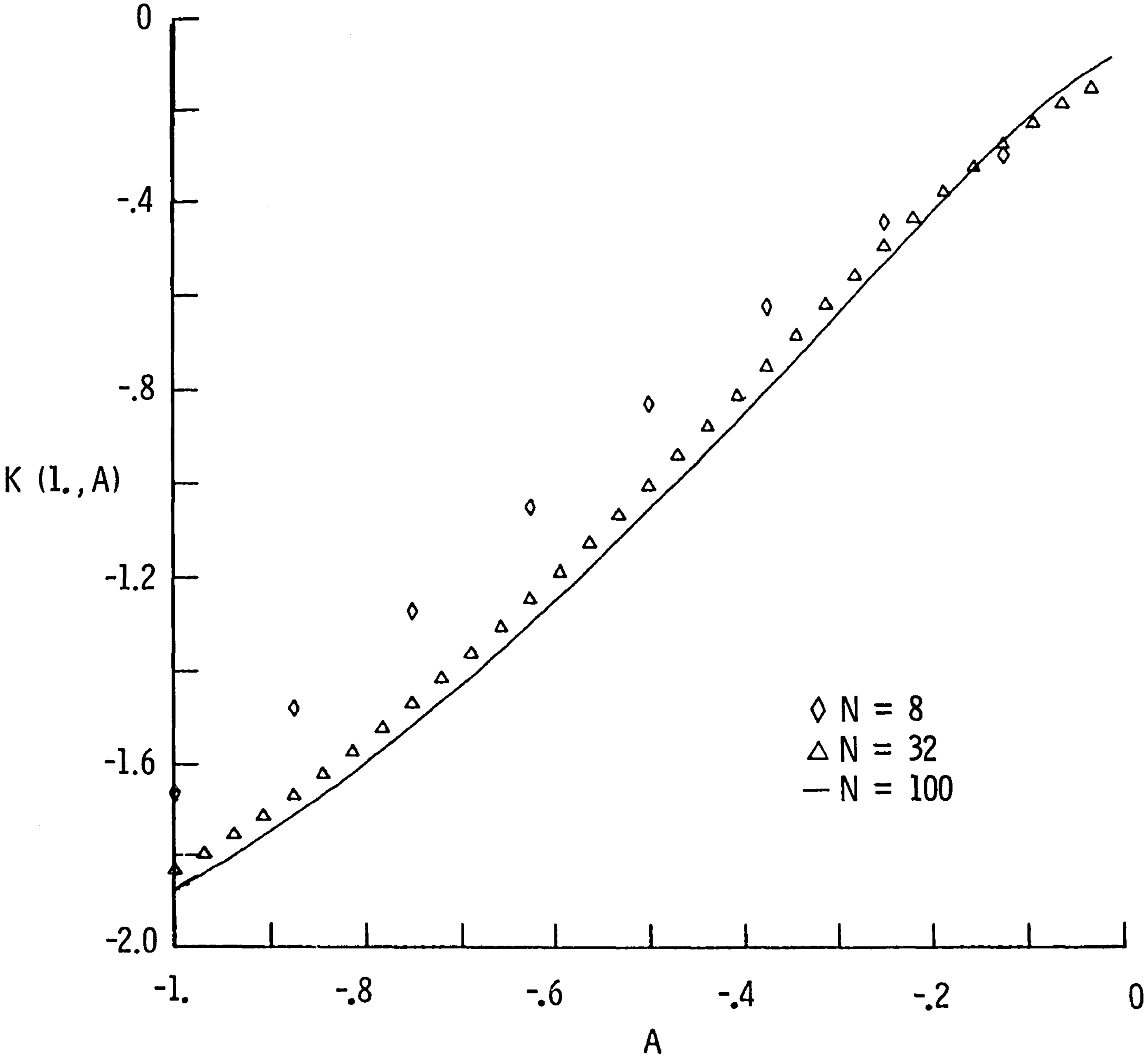

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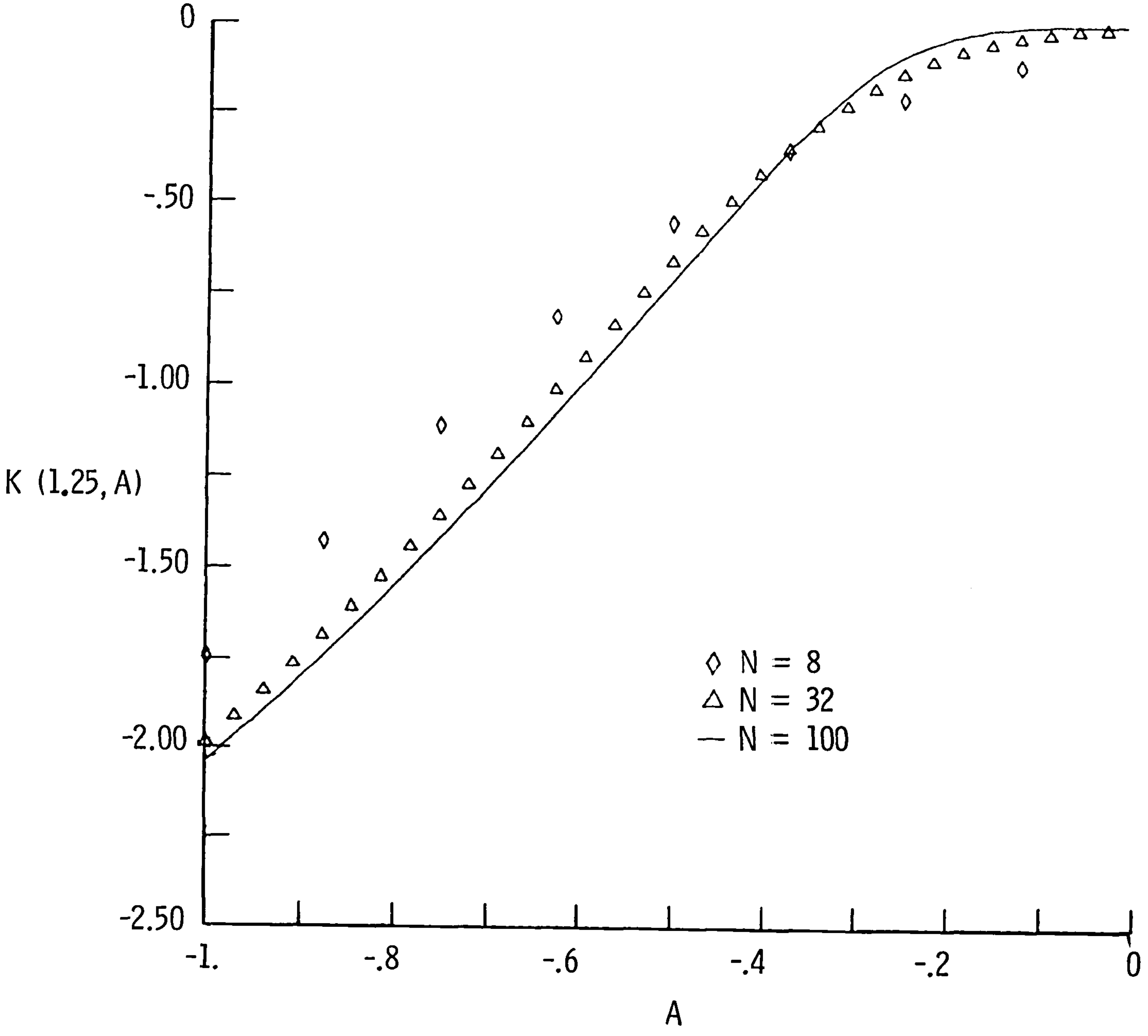

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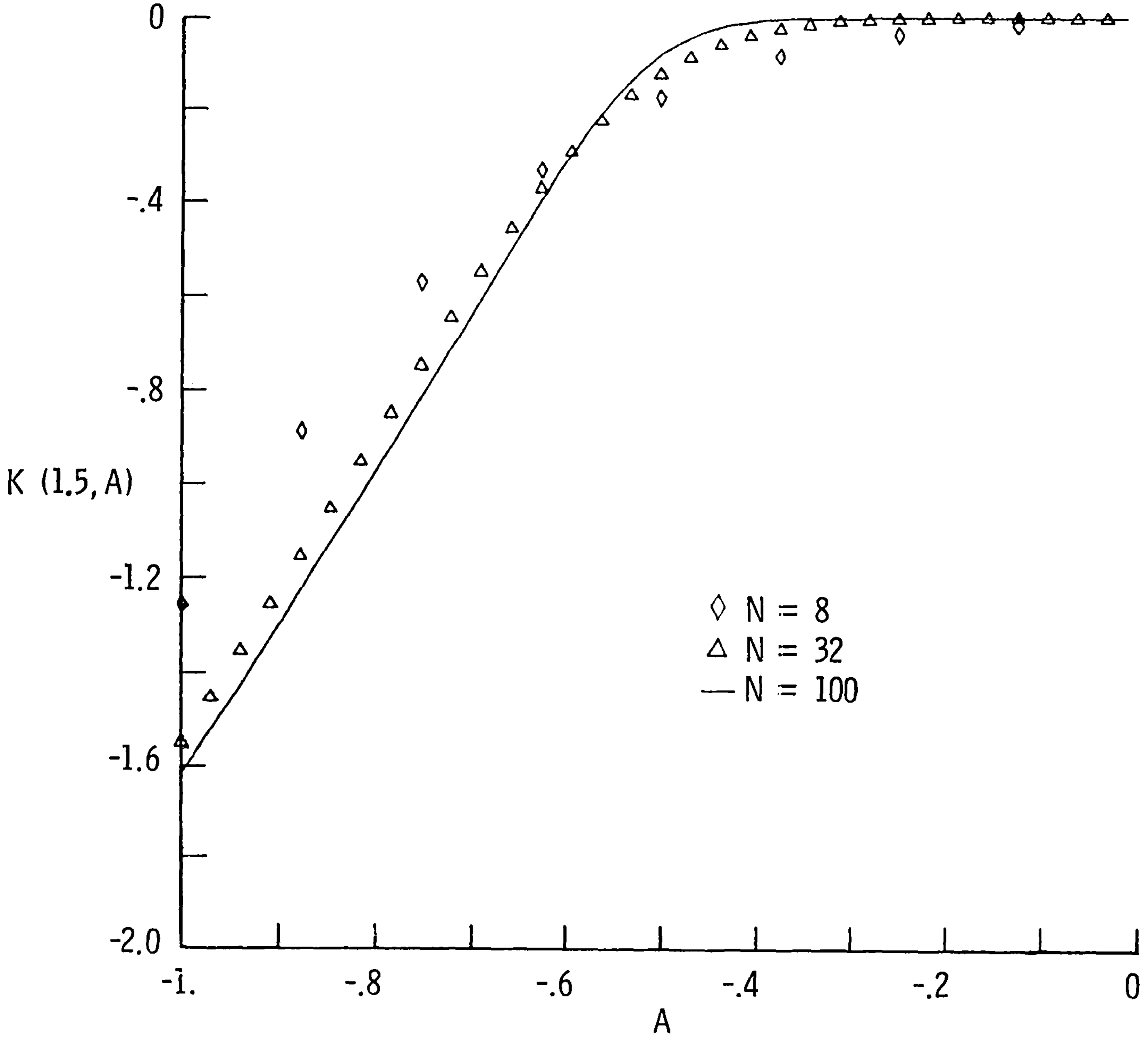

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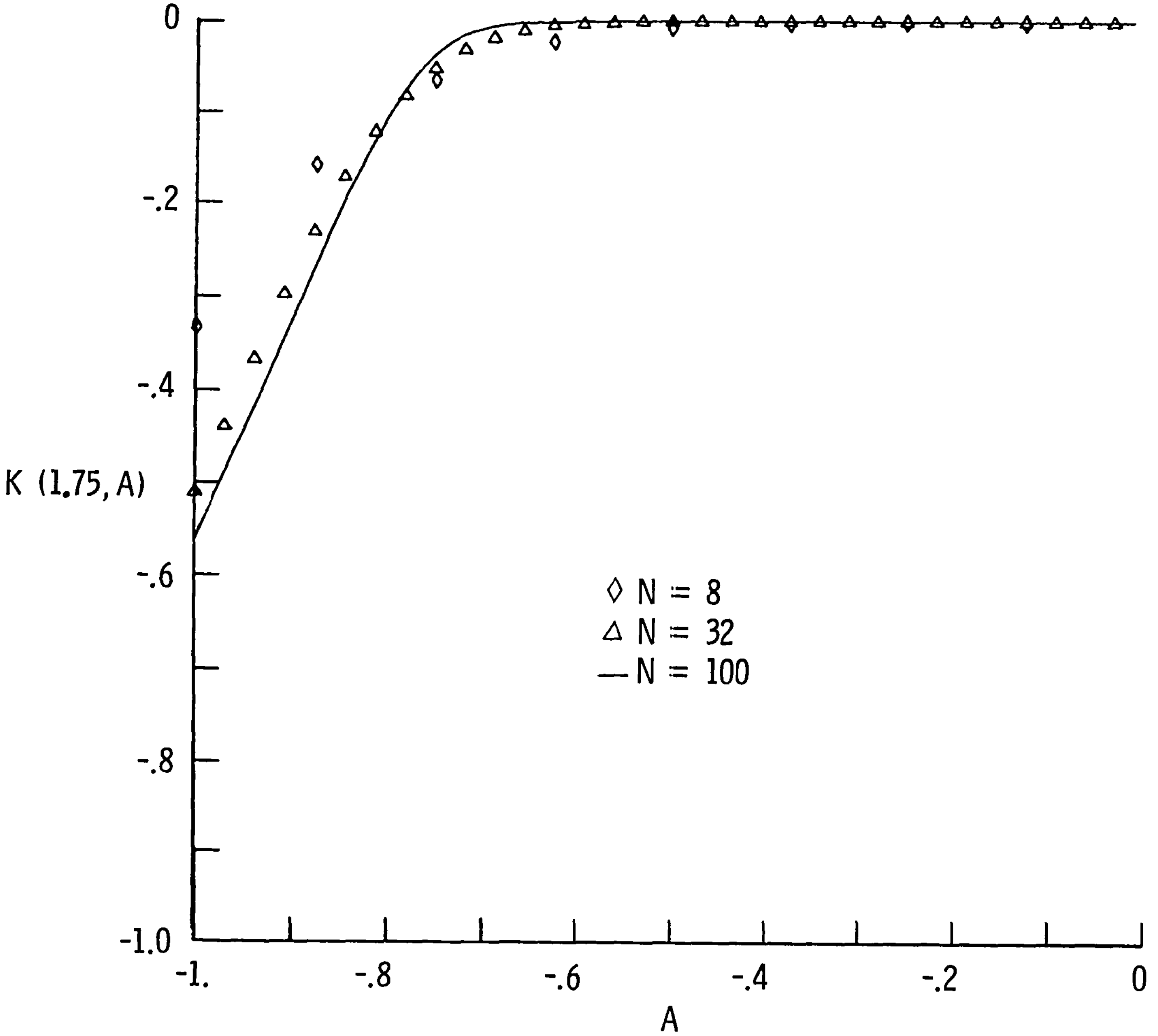

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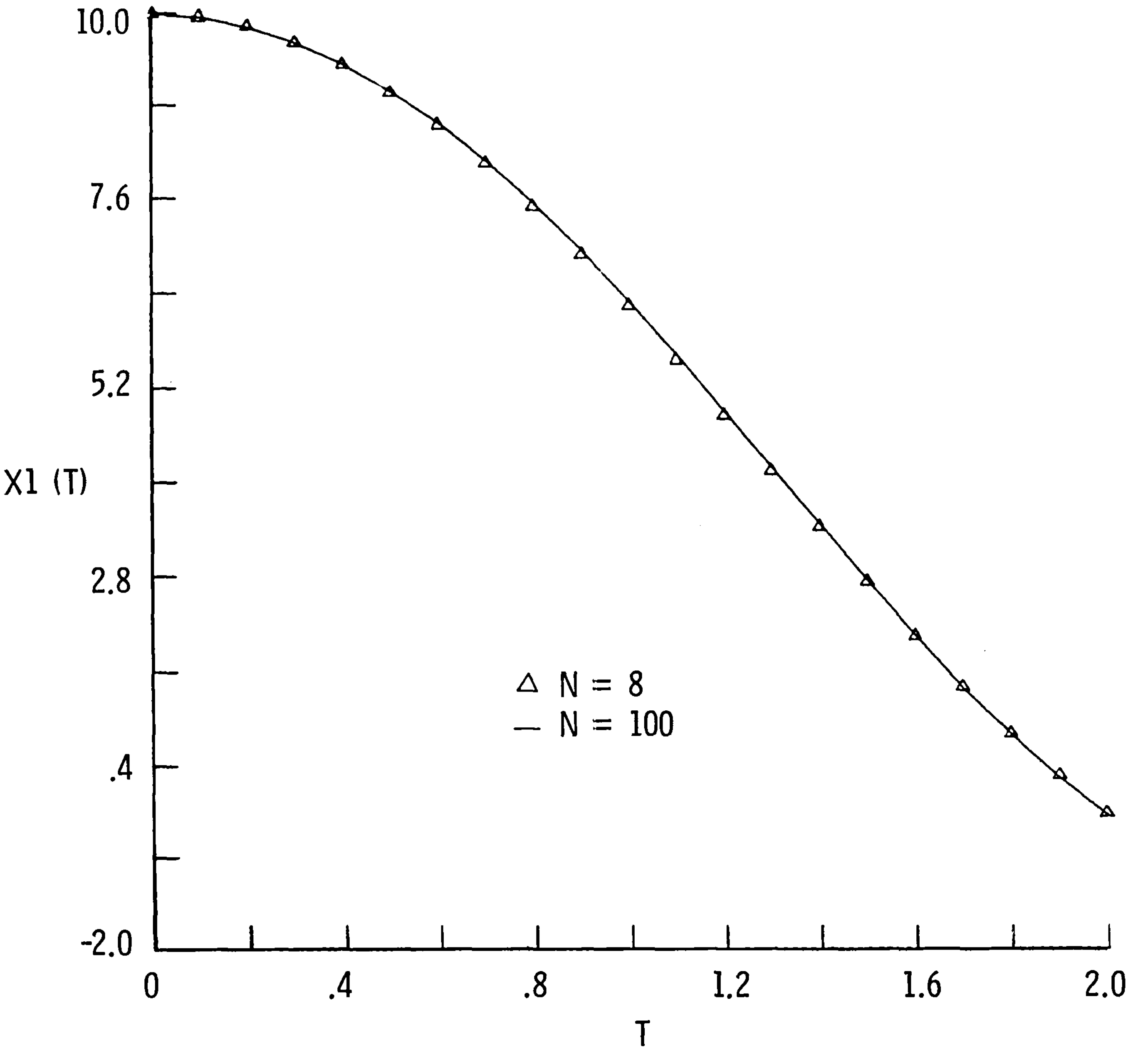

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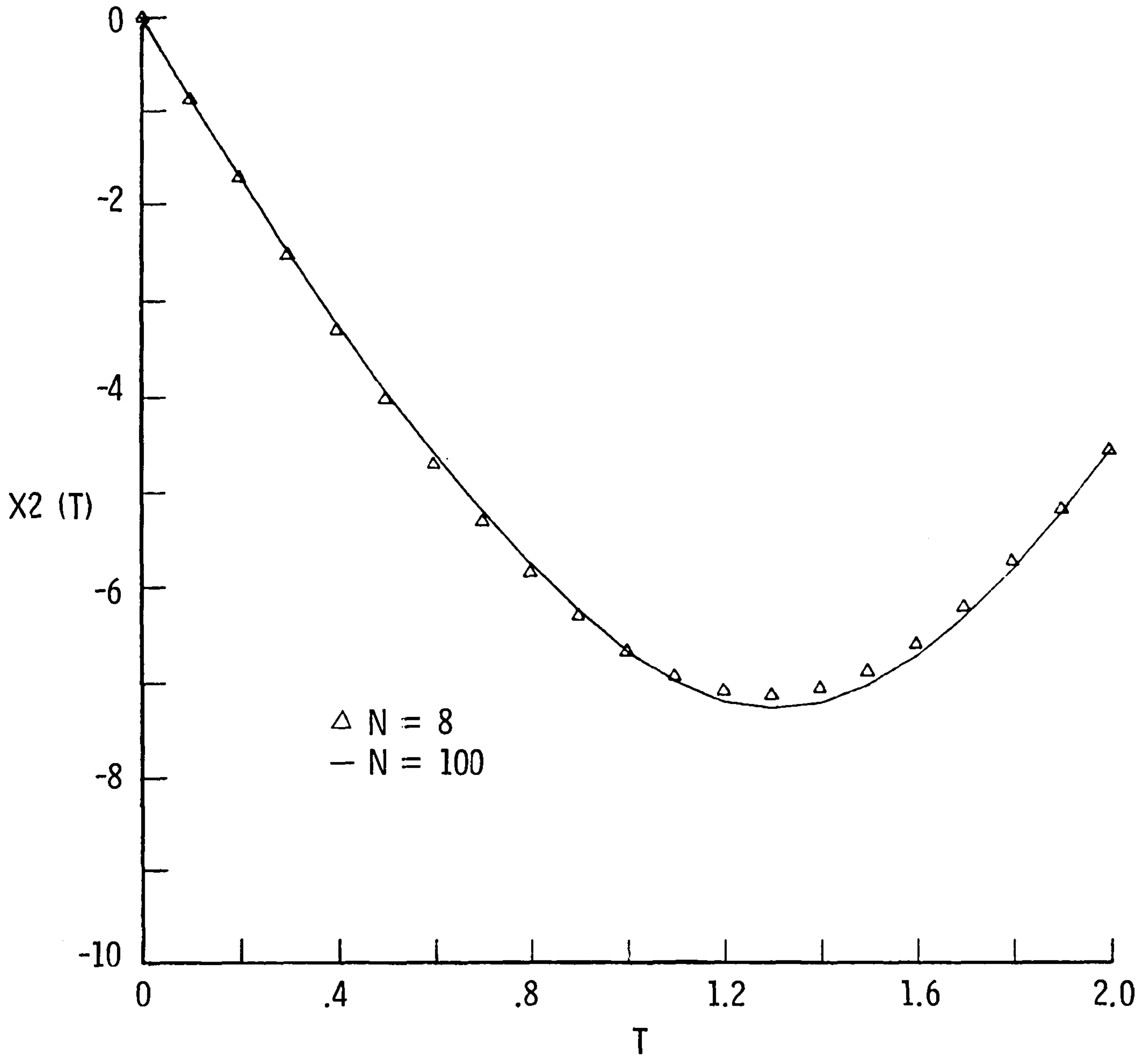

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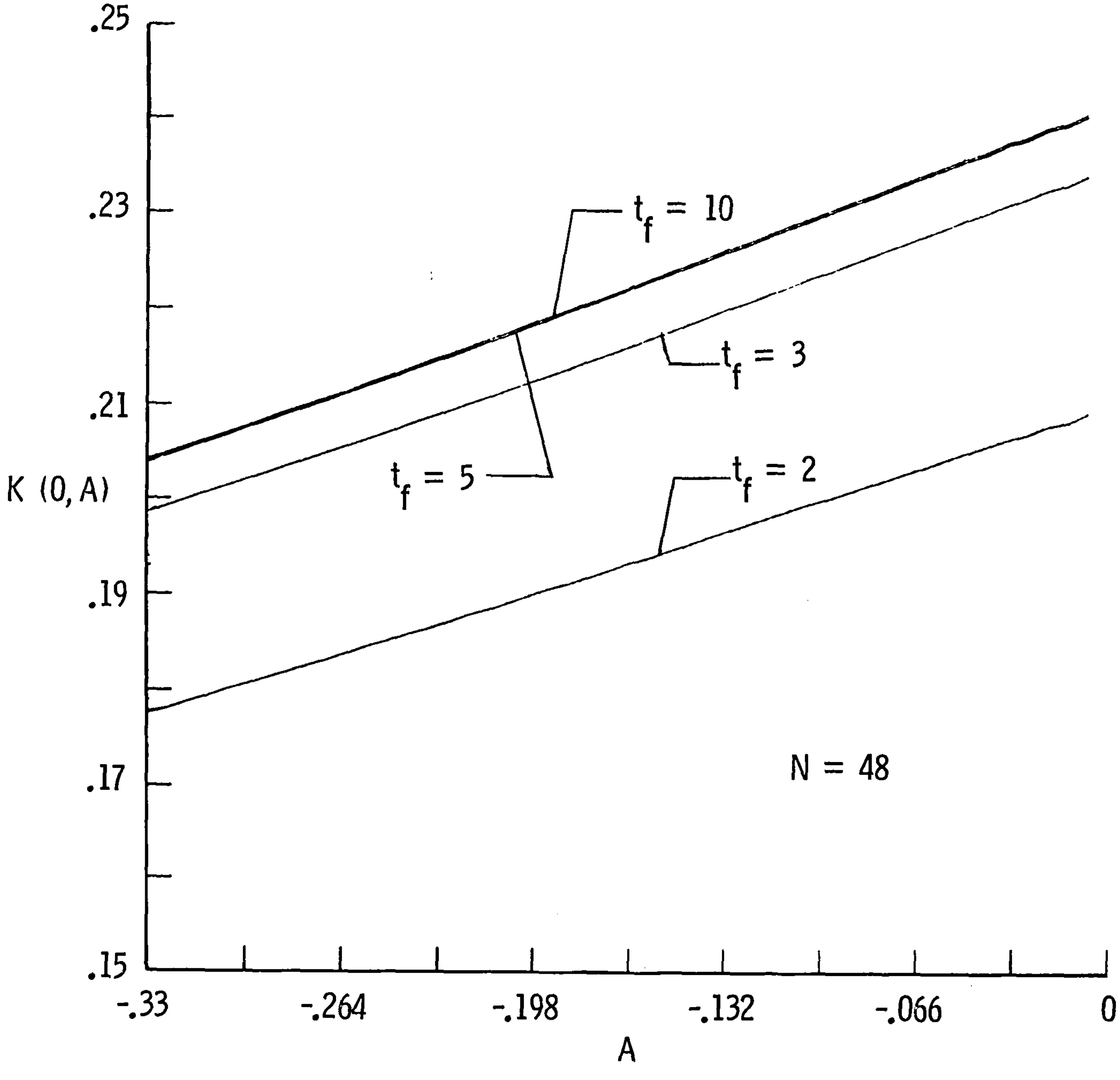

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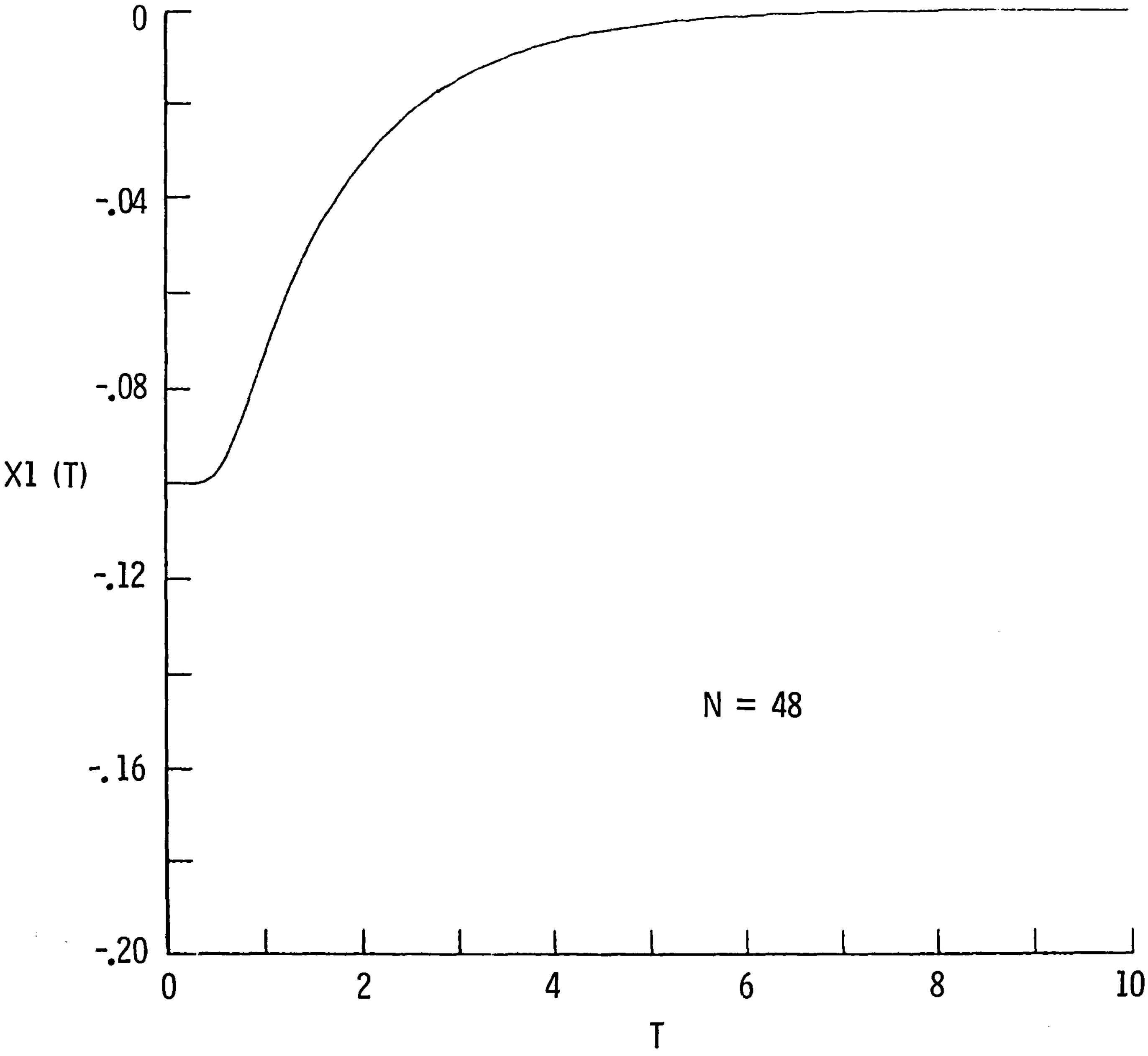

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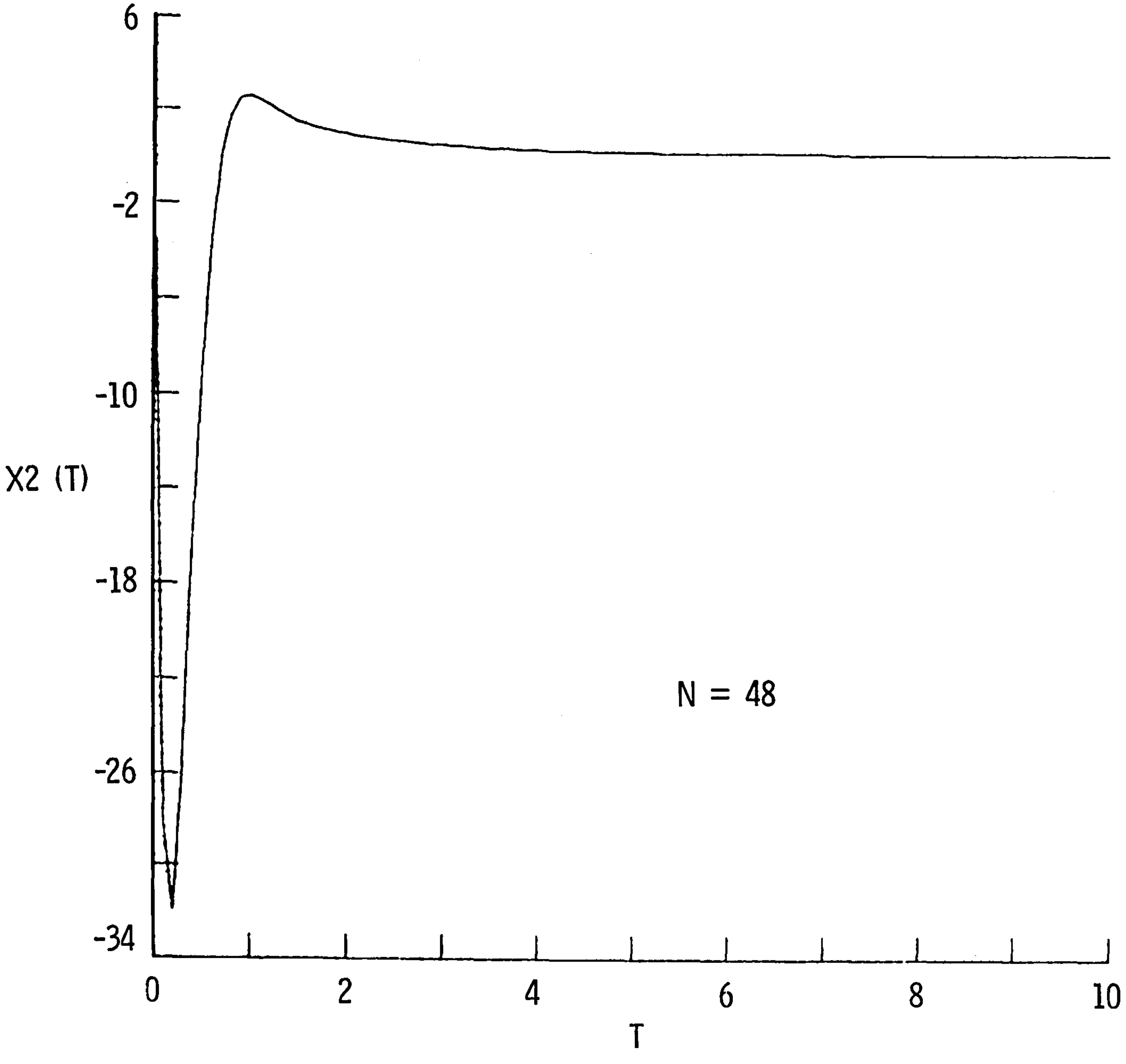

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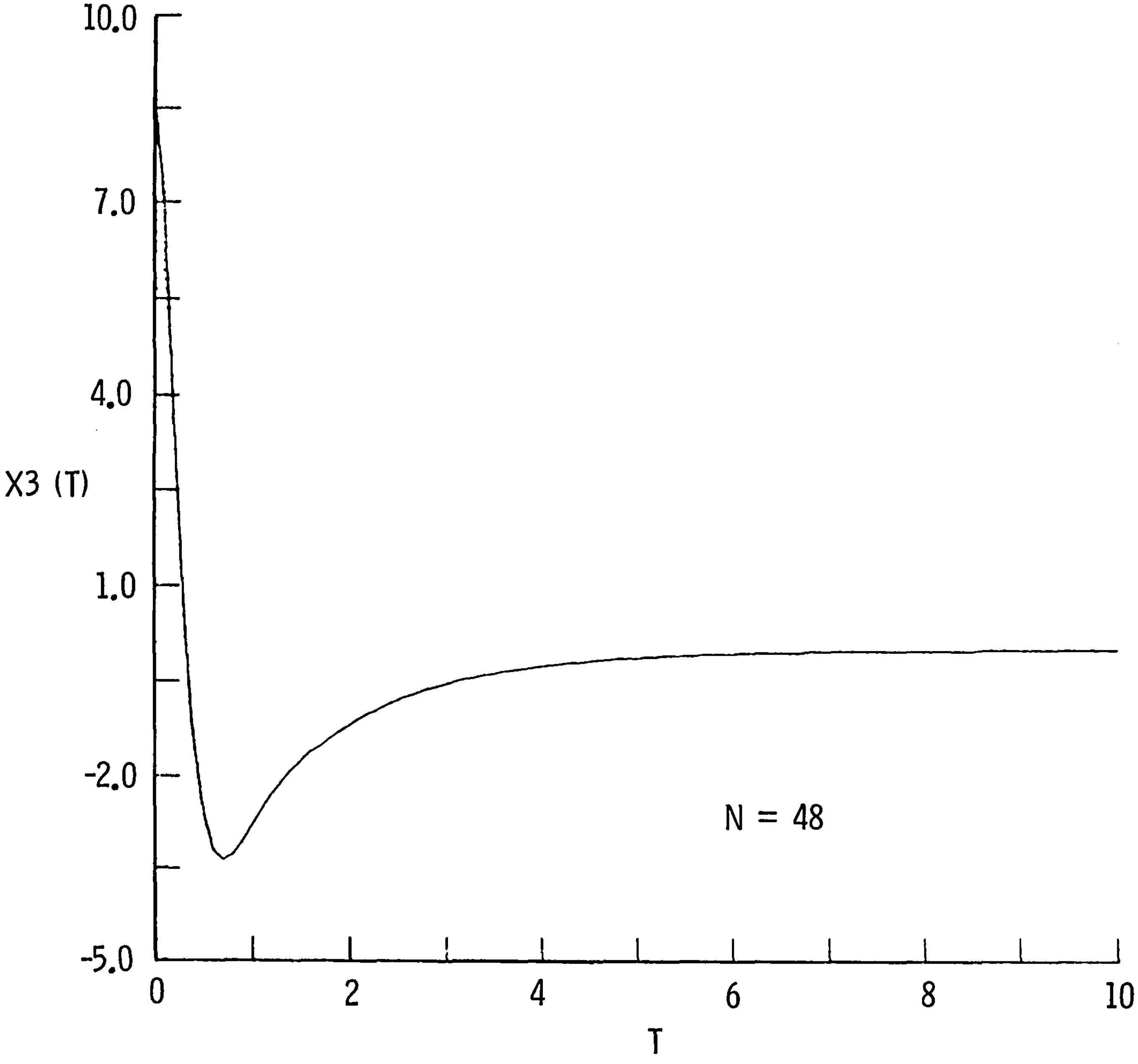

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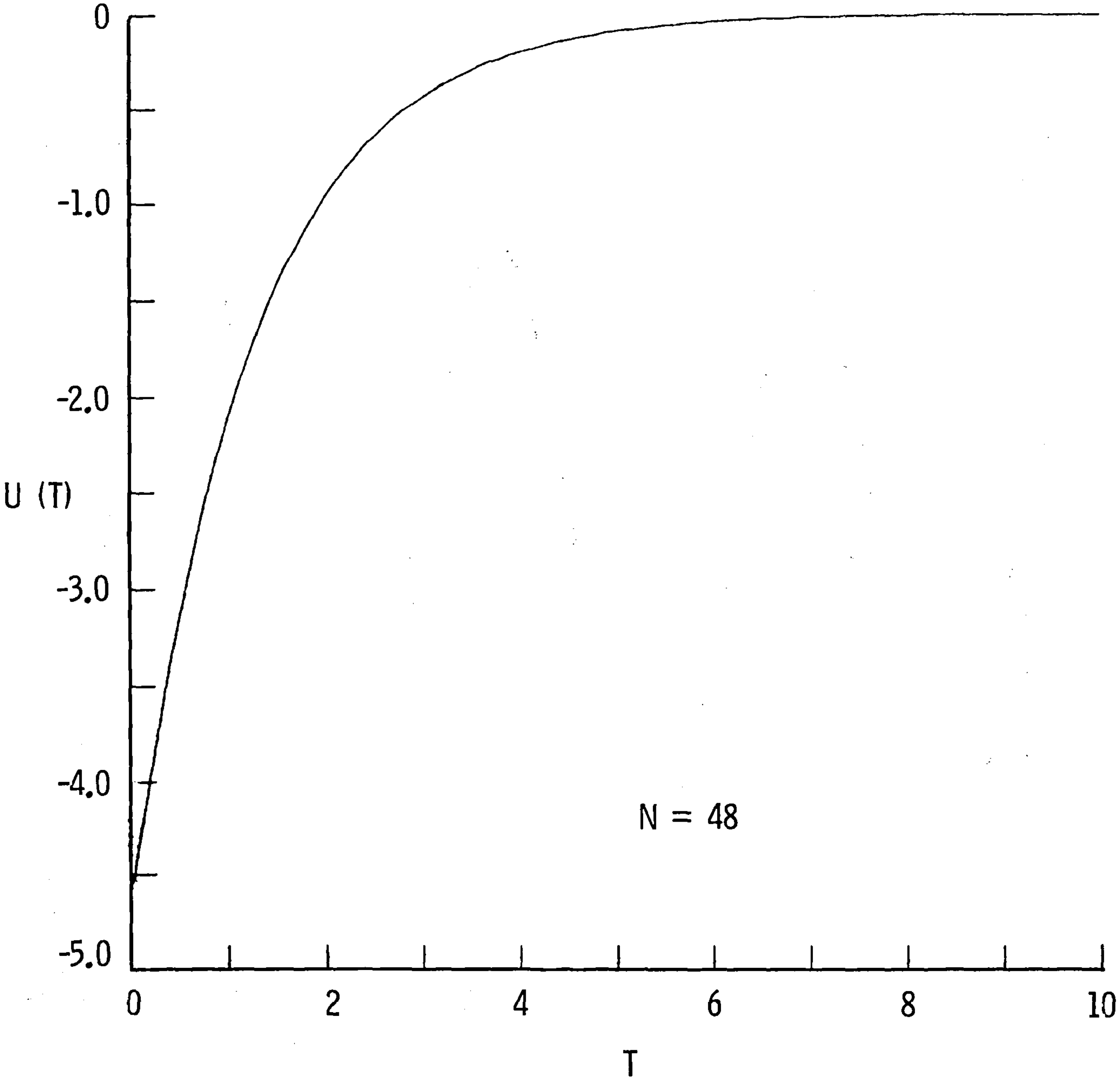

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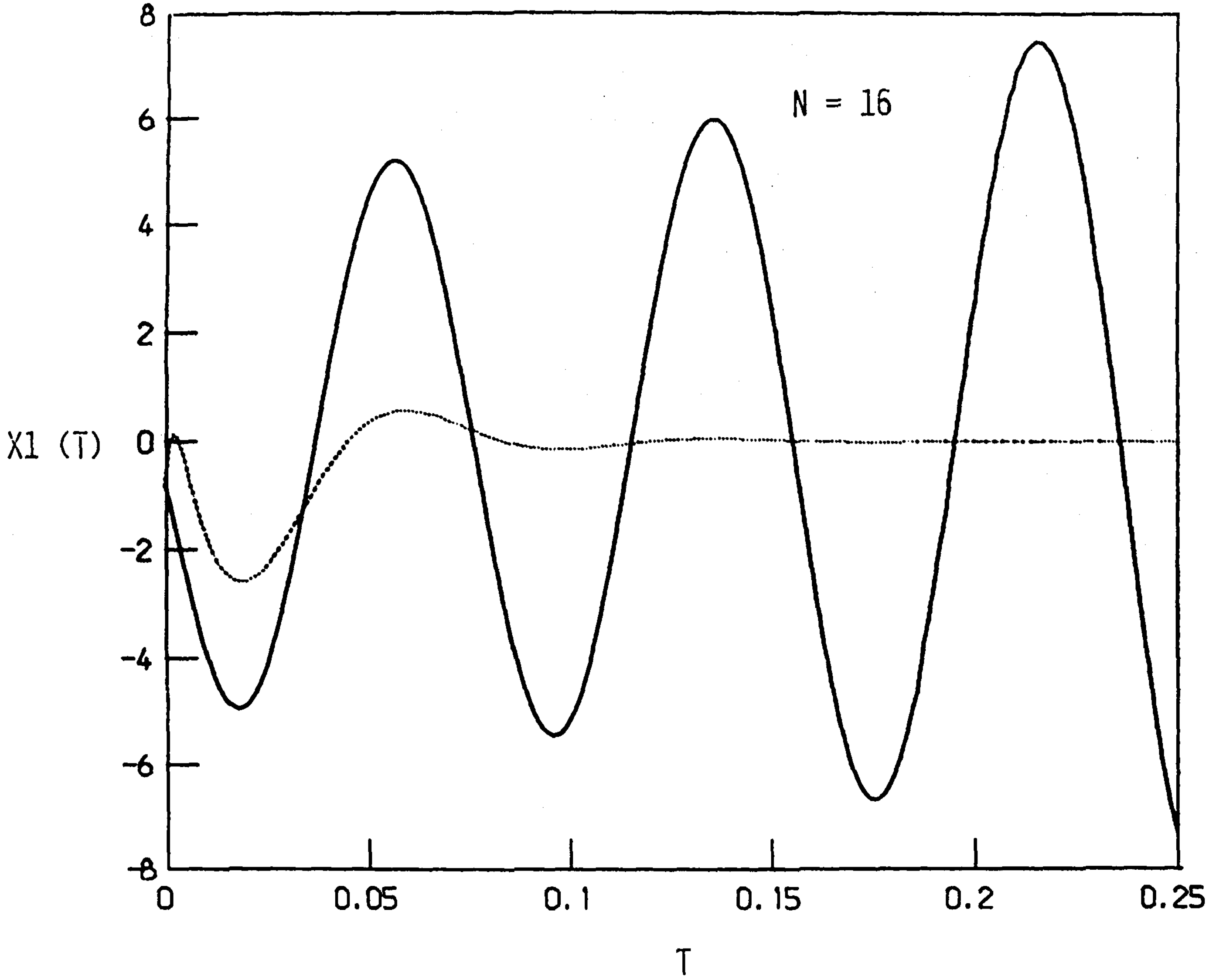

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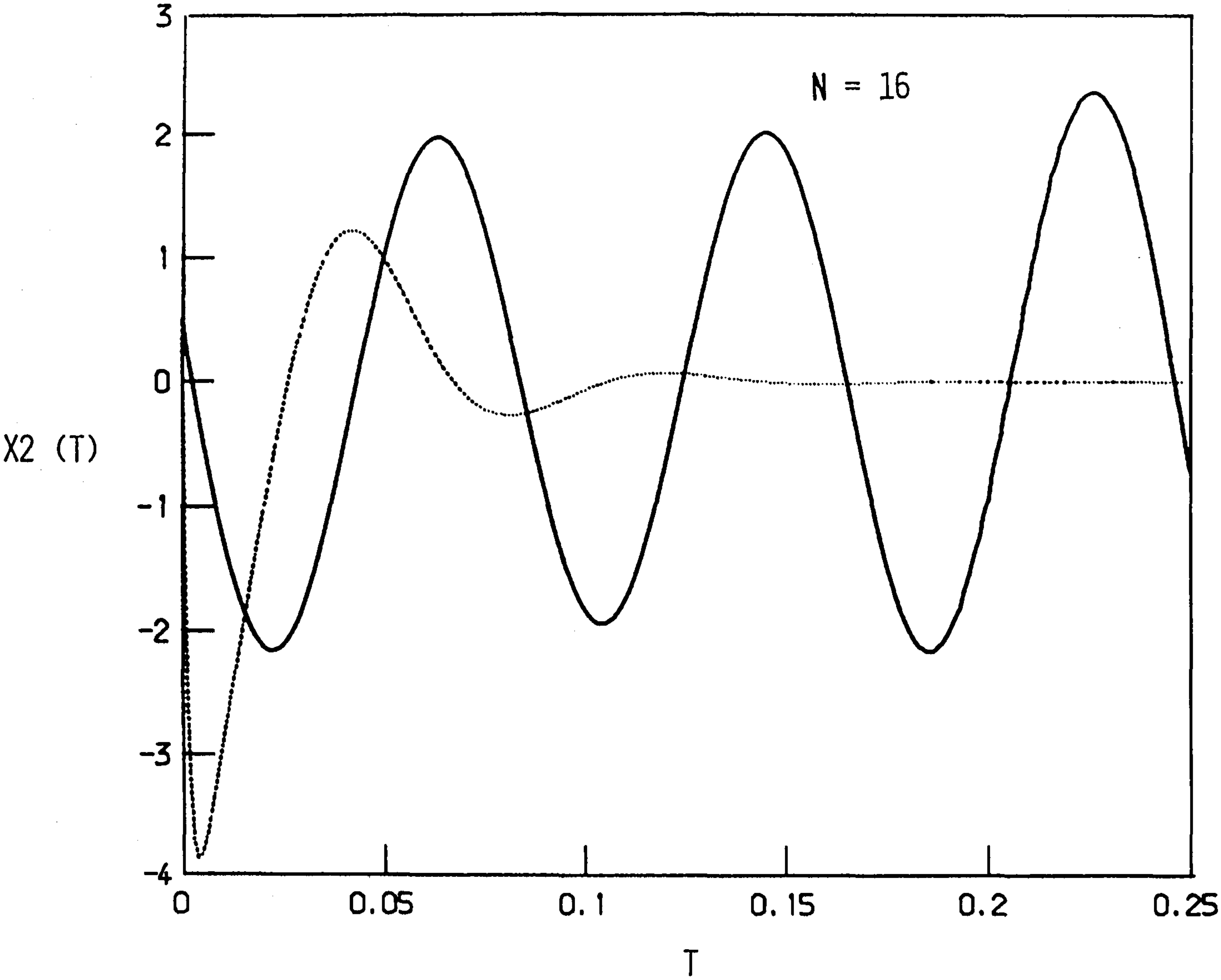

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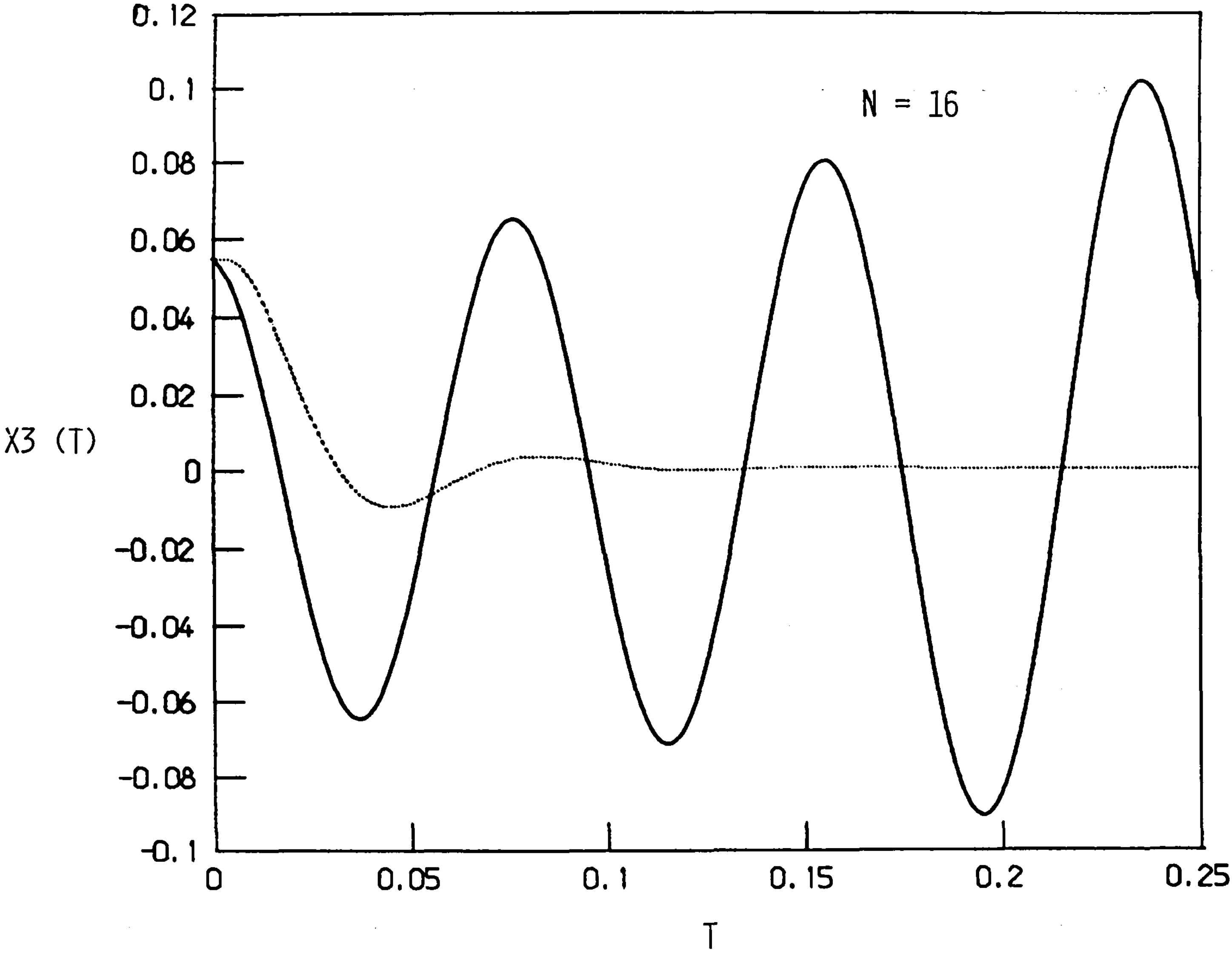
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X3 (T)
T
0.12
0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
0
0.05
0.1
0.15
0.2
0.25

FIGURE 5.21

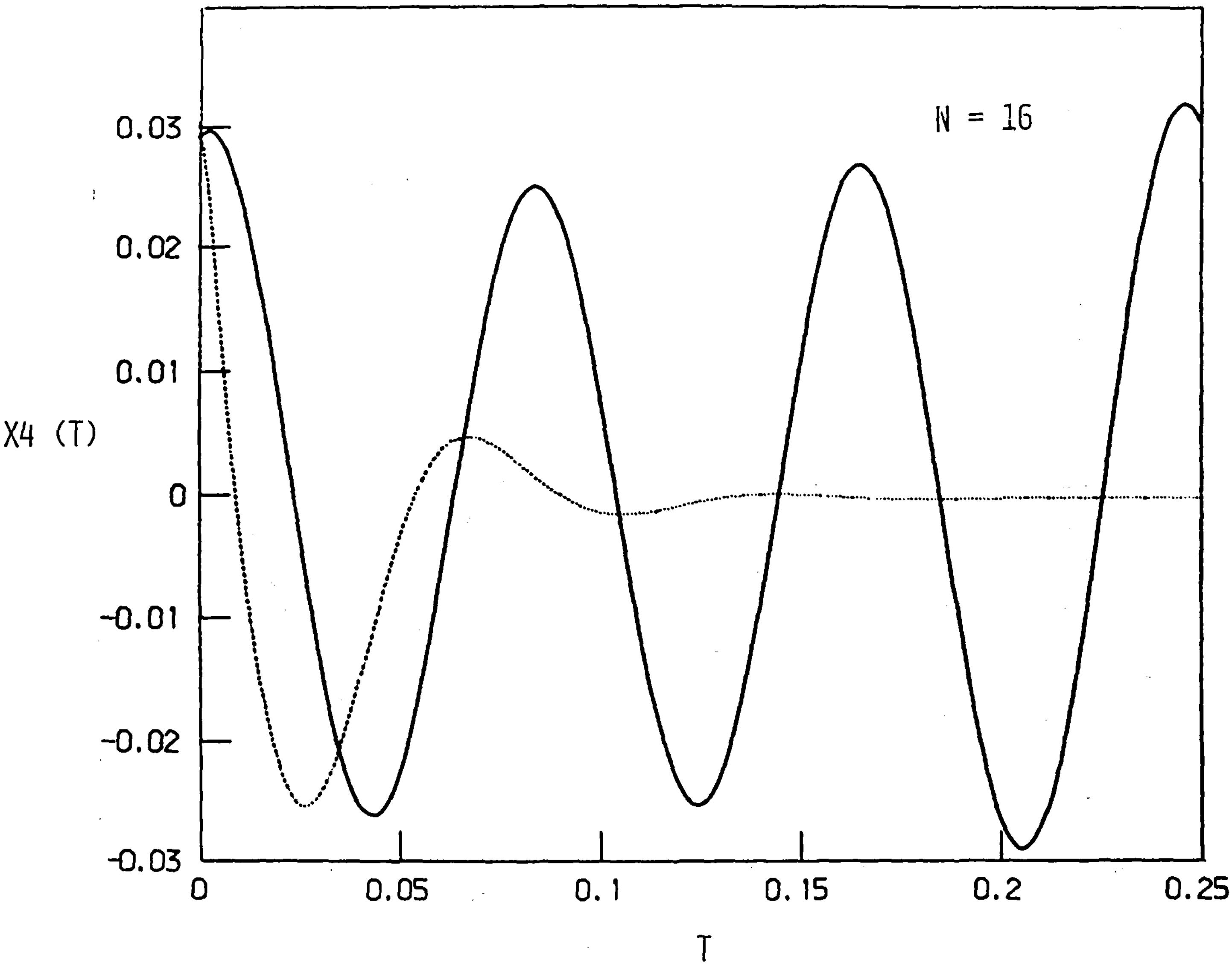

FIGURE 5.22

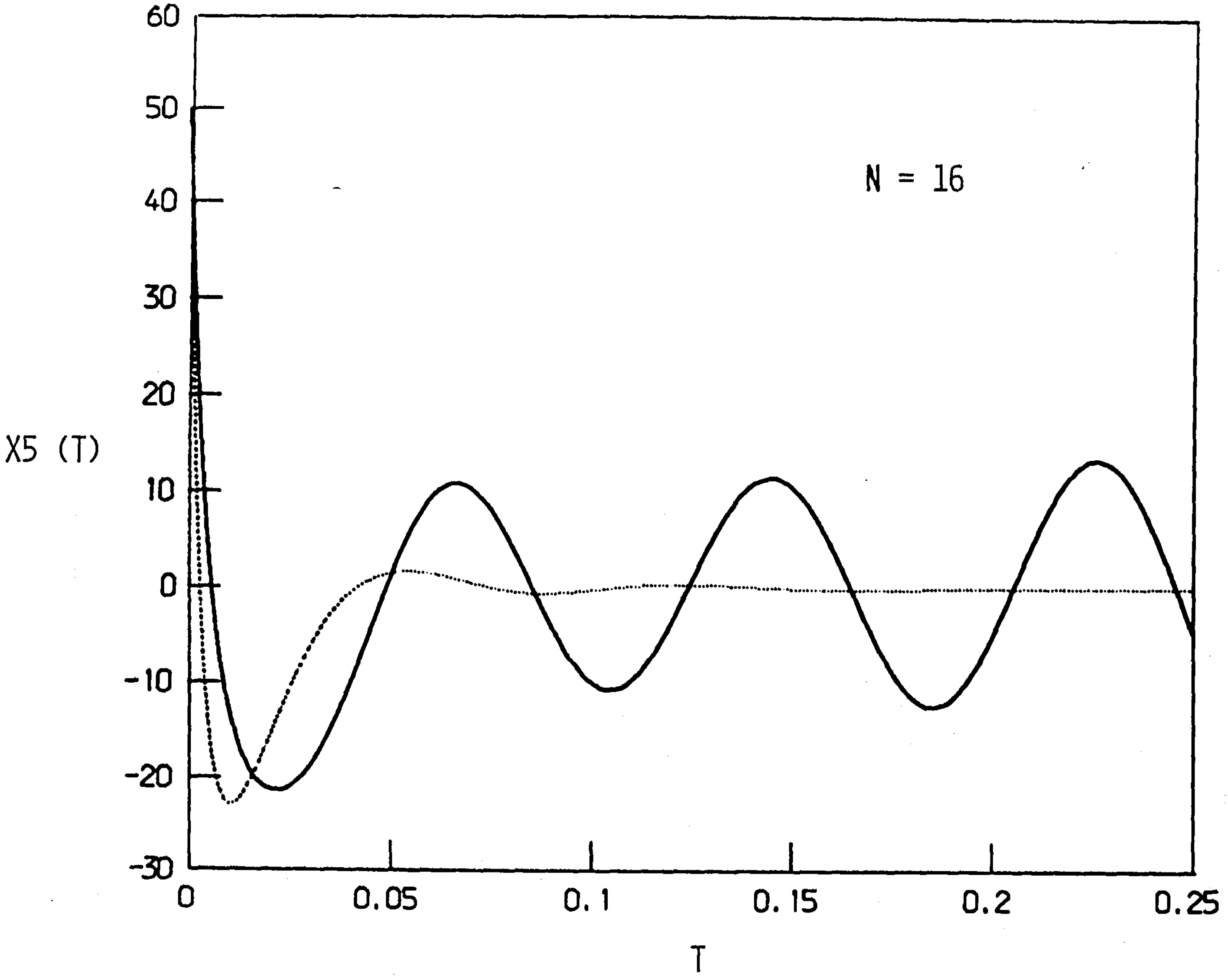

FIGURE 5.23

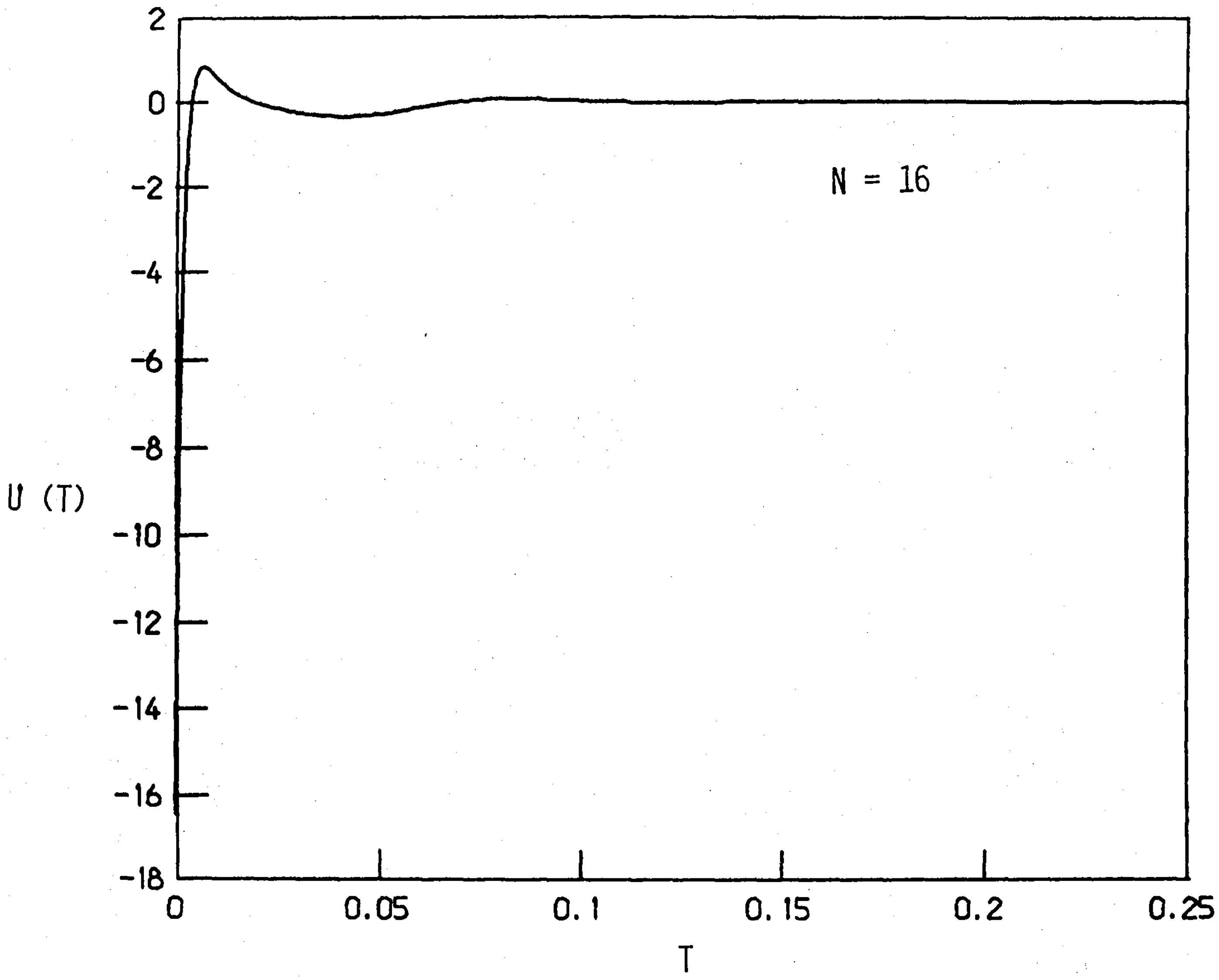

FIGURE 5.24

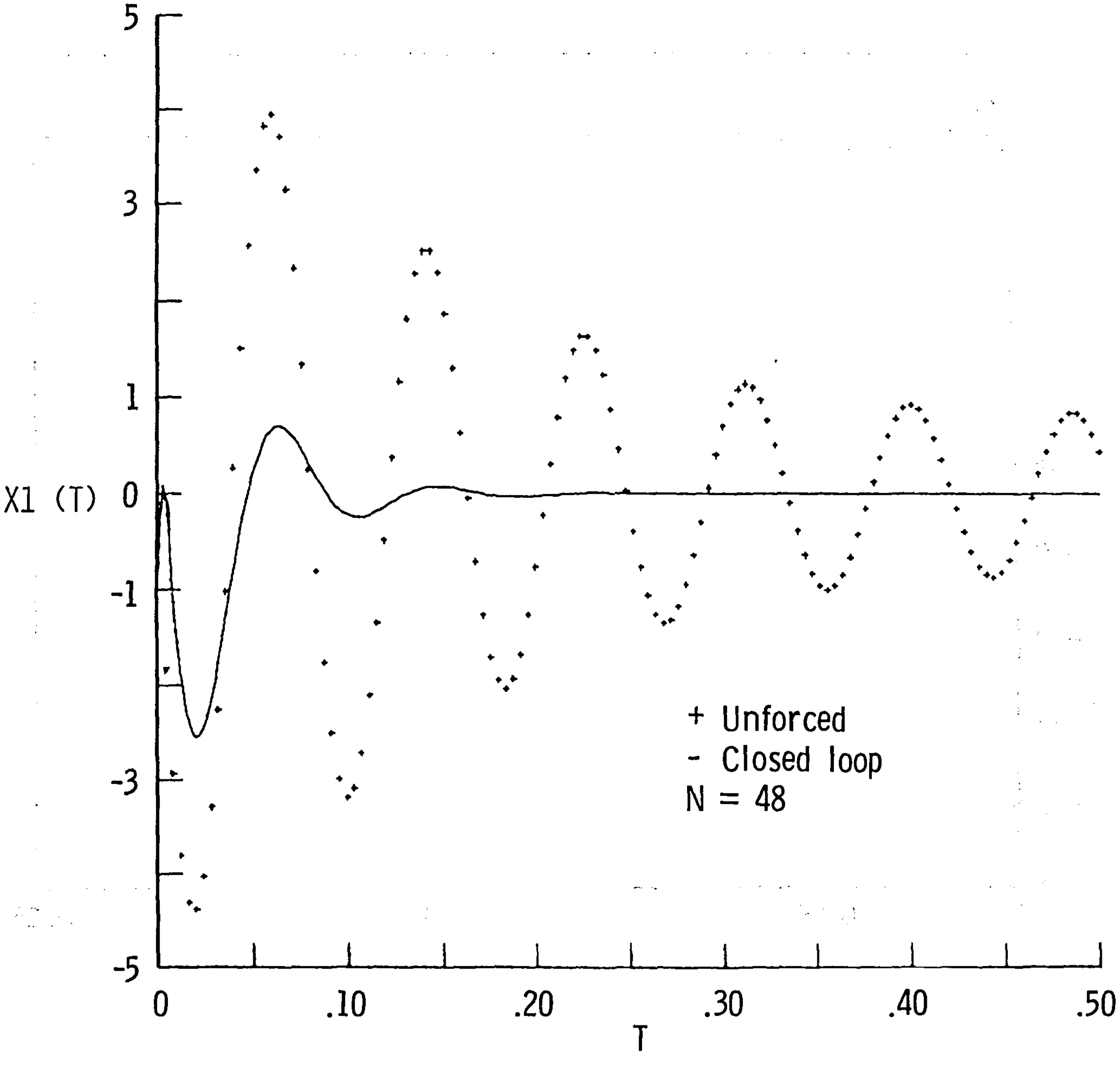

FIGURE 5.25

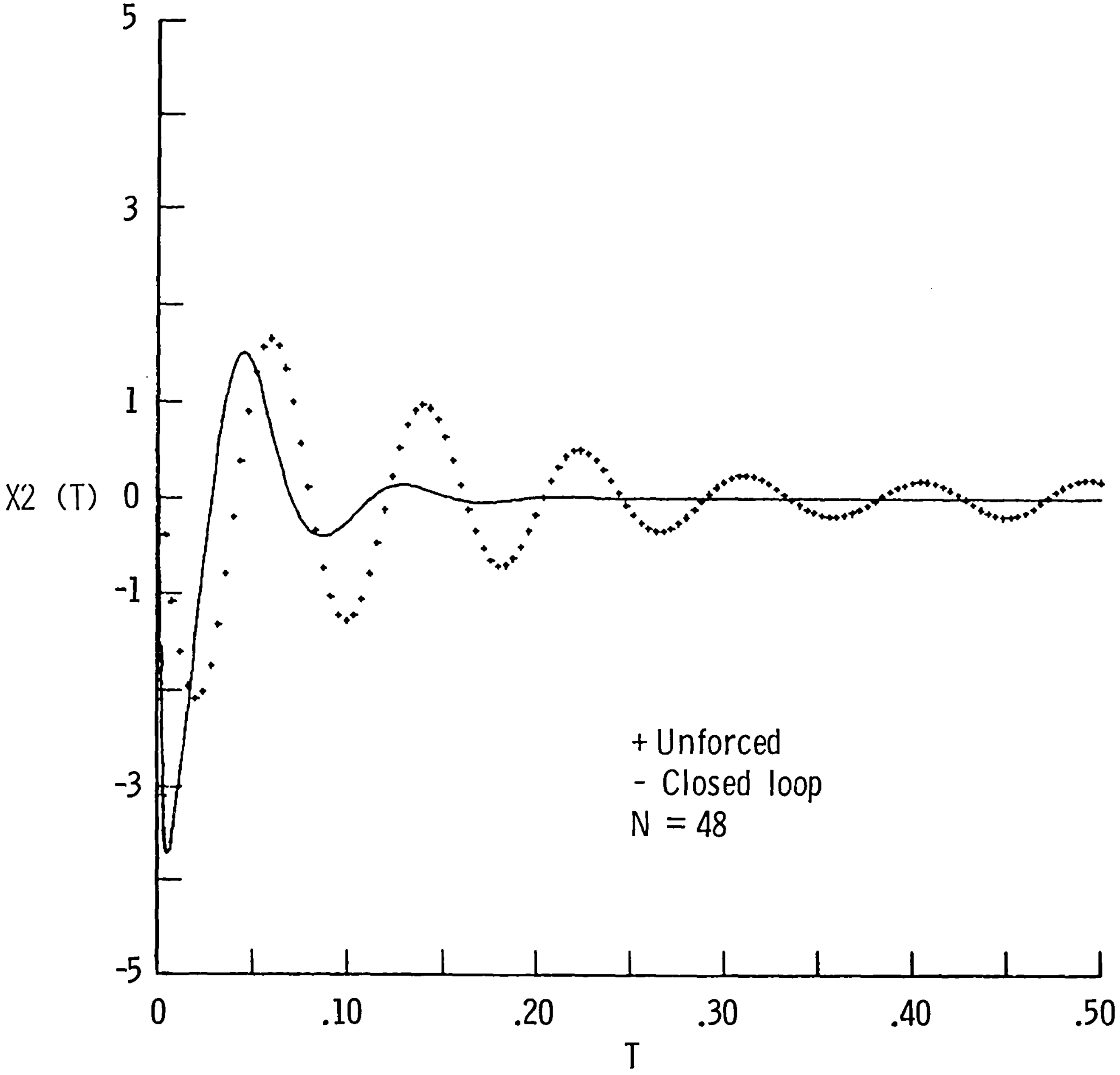

FIGURE 5.26

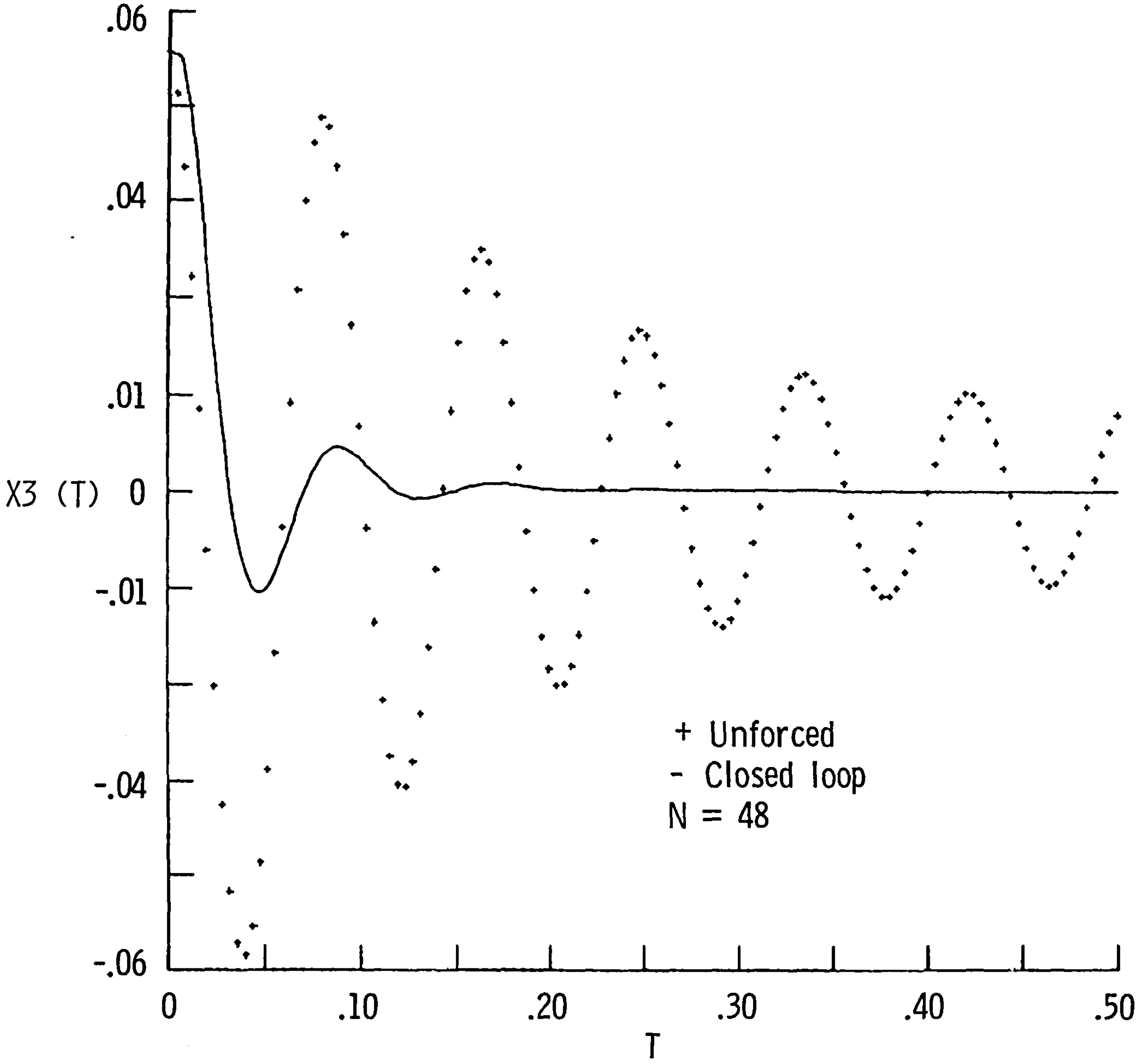

FIGURE 5.27

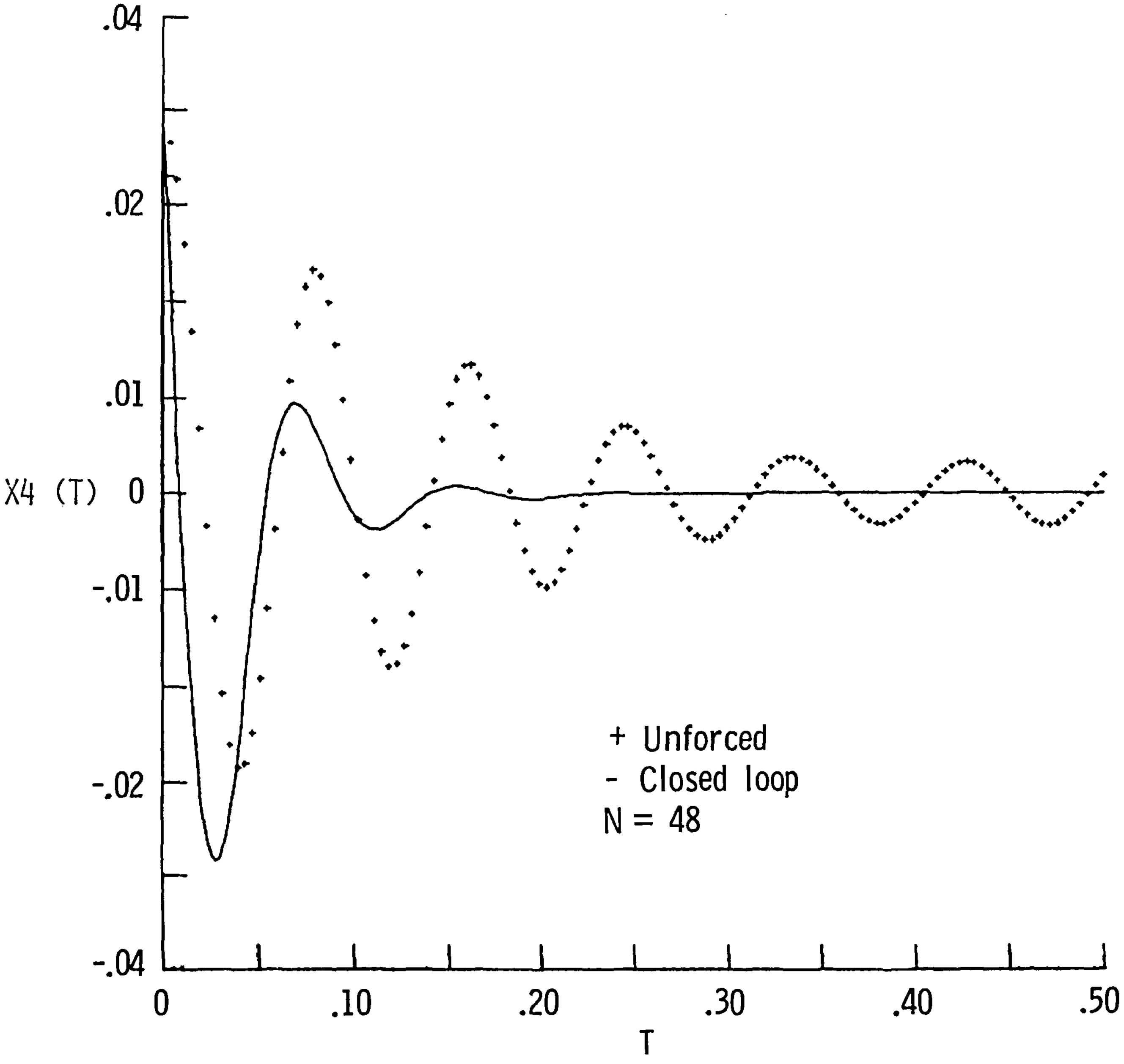

FIGURE 5.28

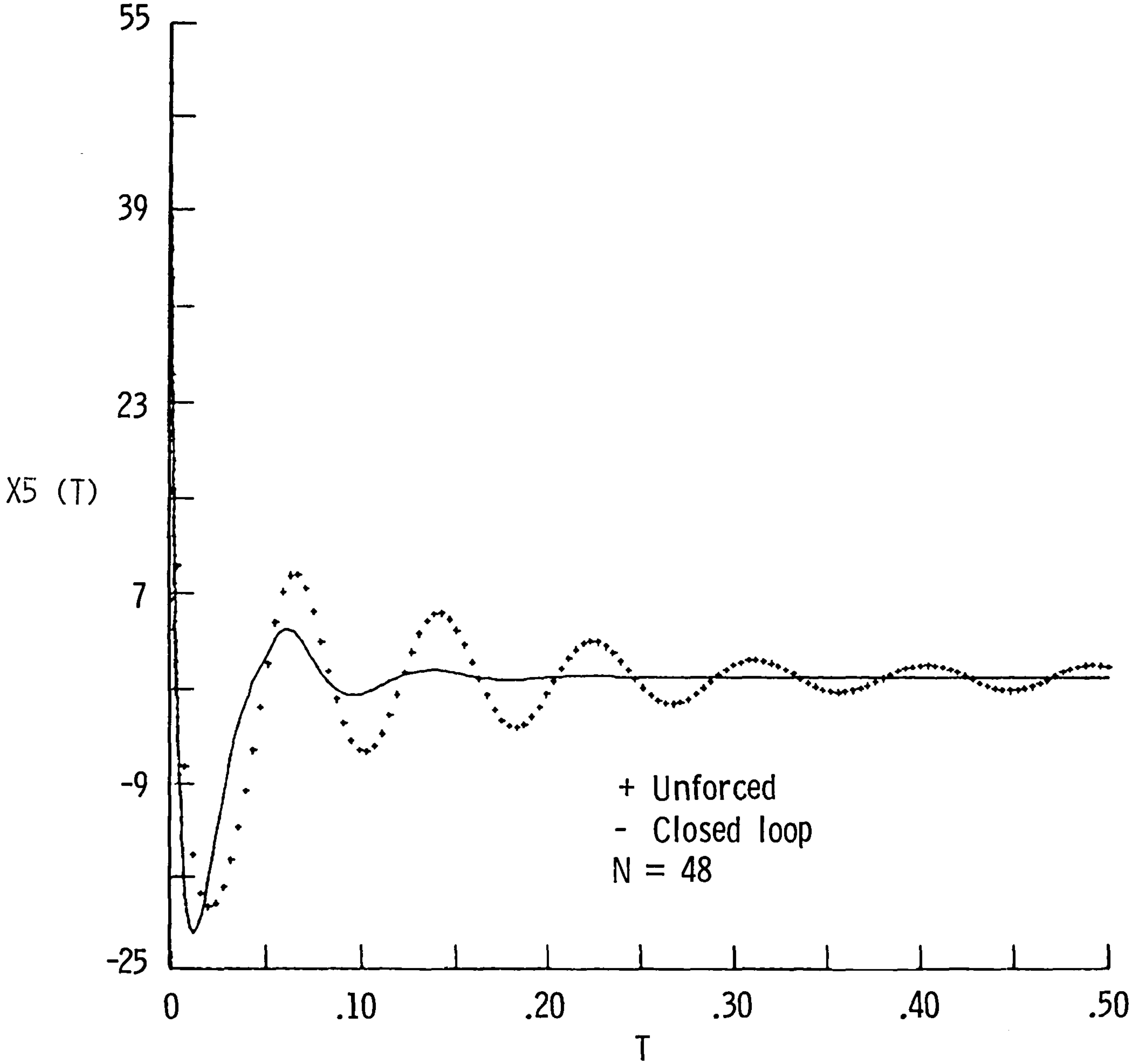

FIGURE 5.29

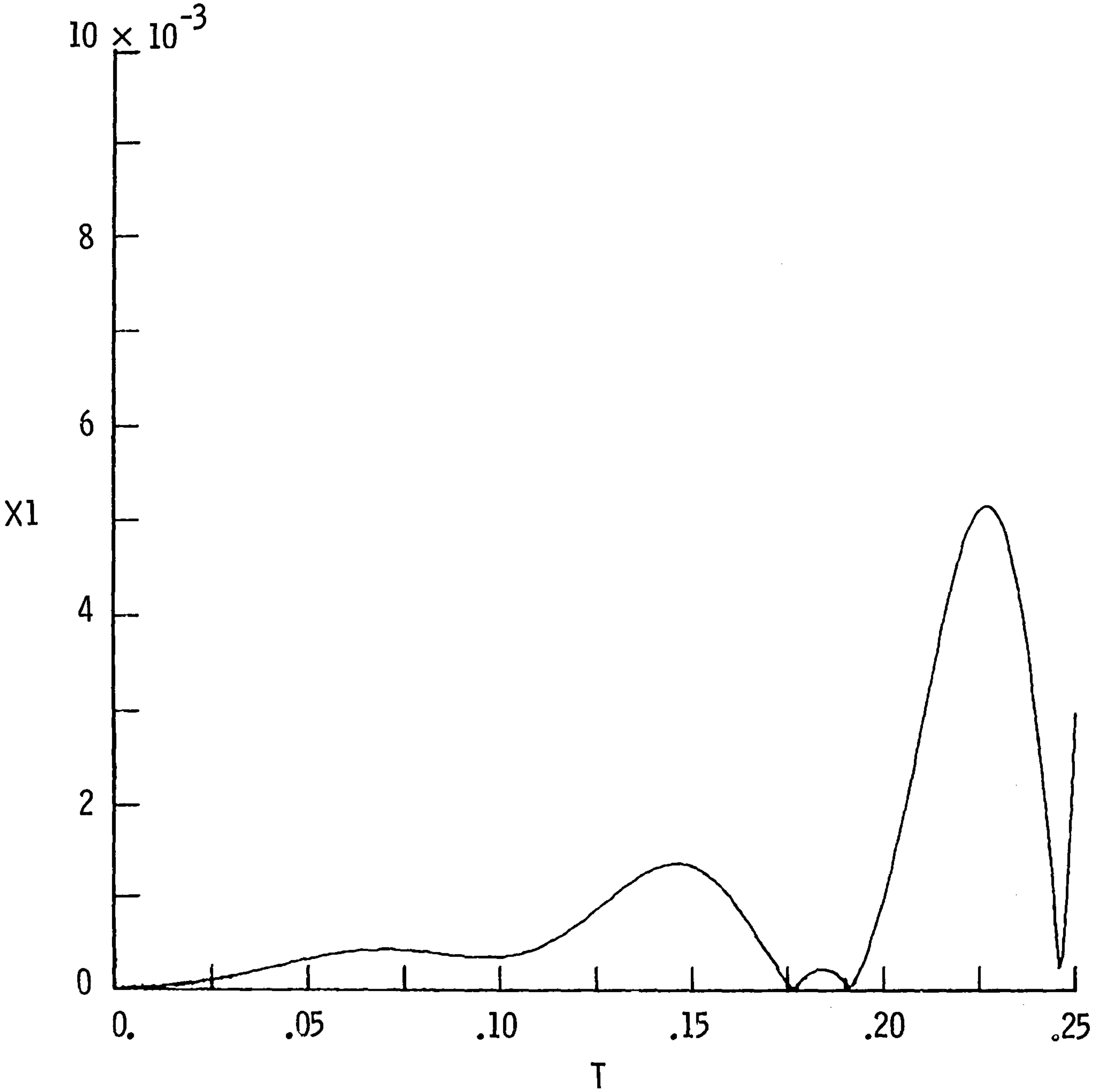

FIGURE 5.30

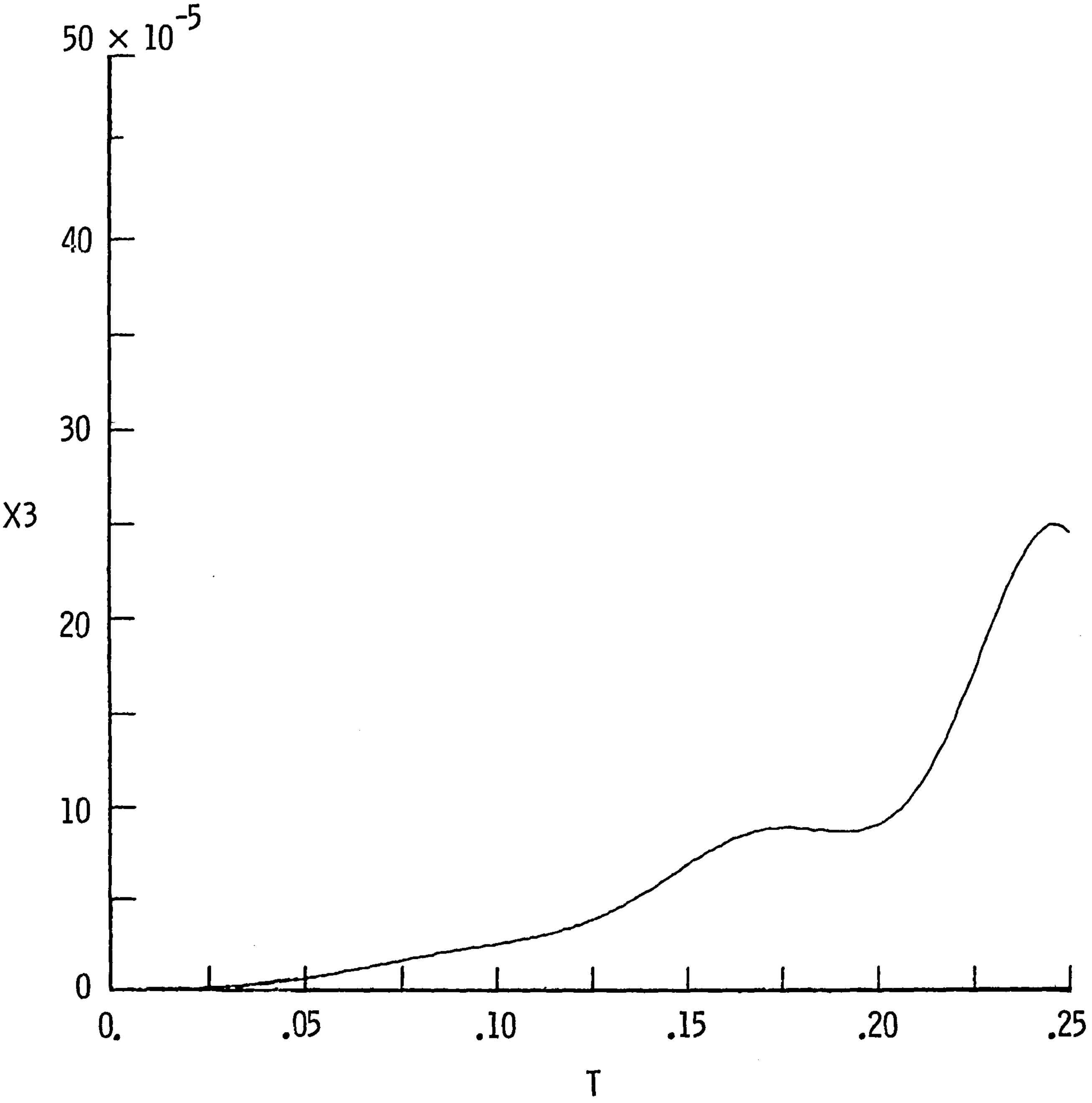

FIGURE 5.31

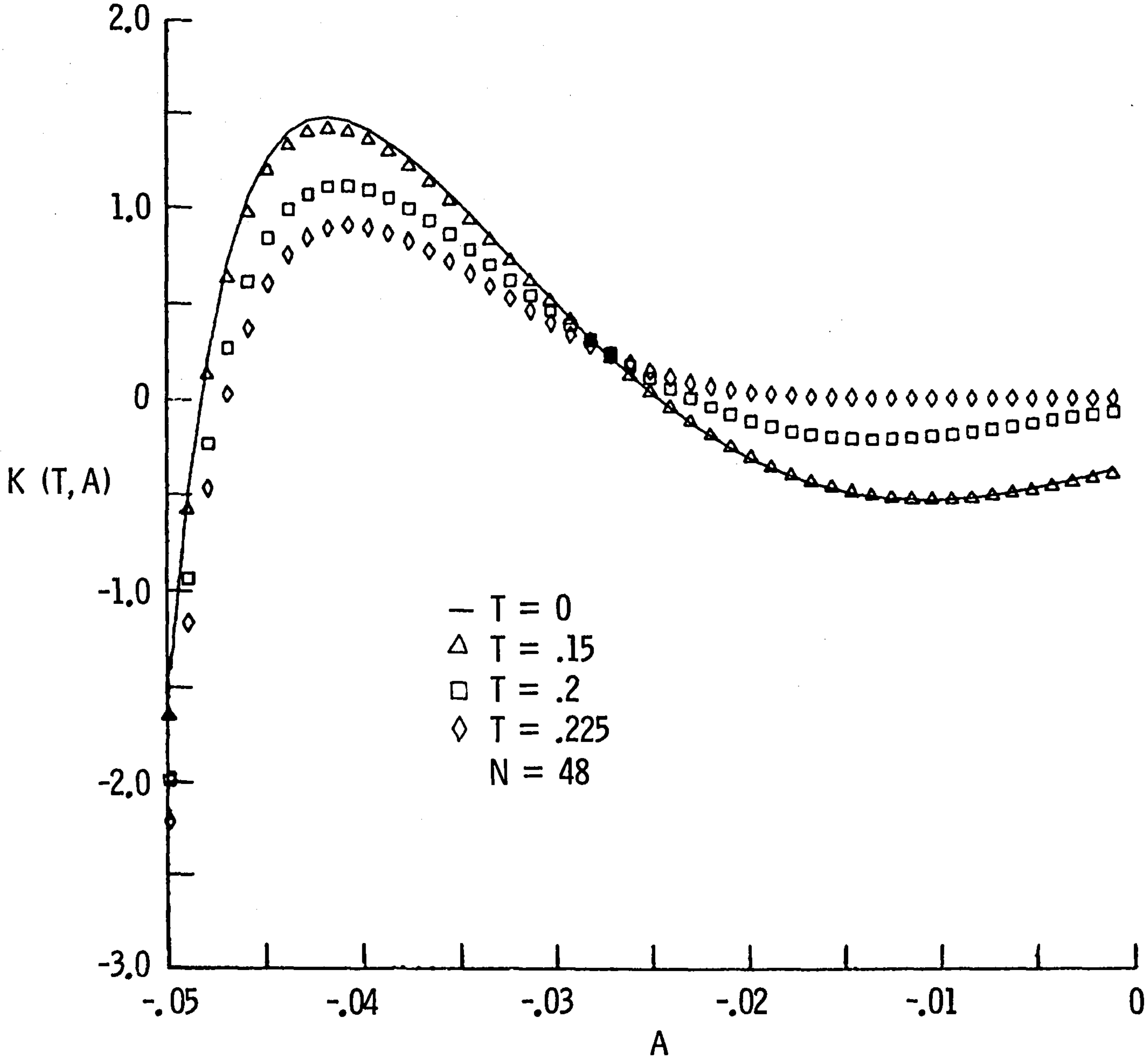

FIGURE 5.32

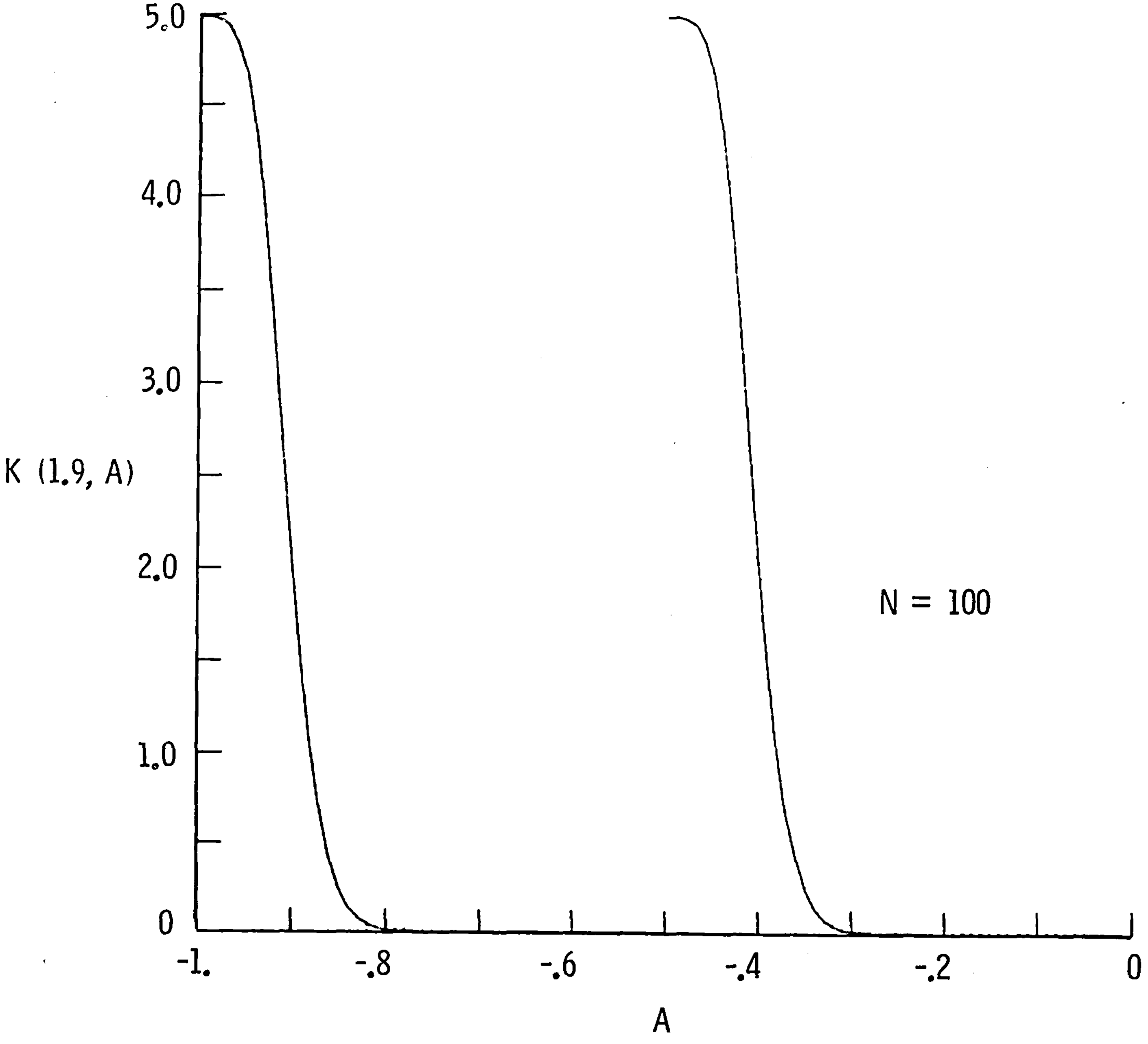

FIGURE 5.33

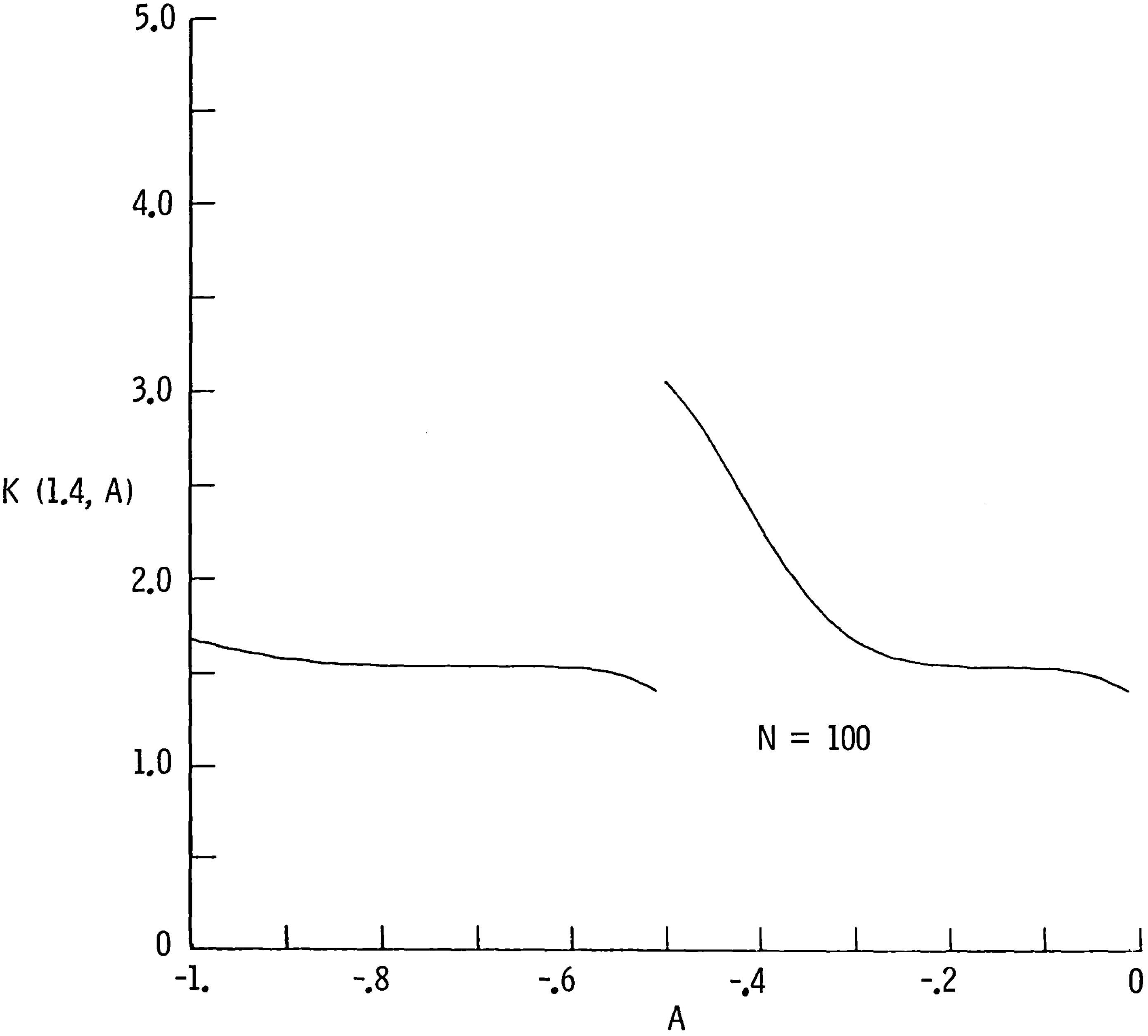

FIGURE 5.34

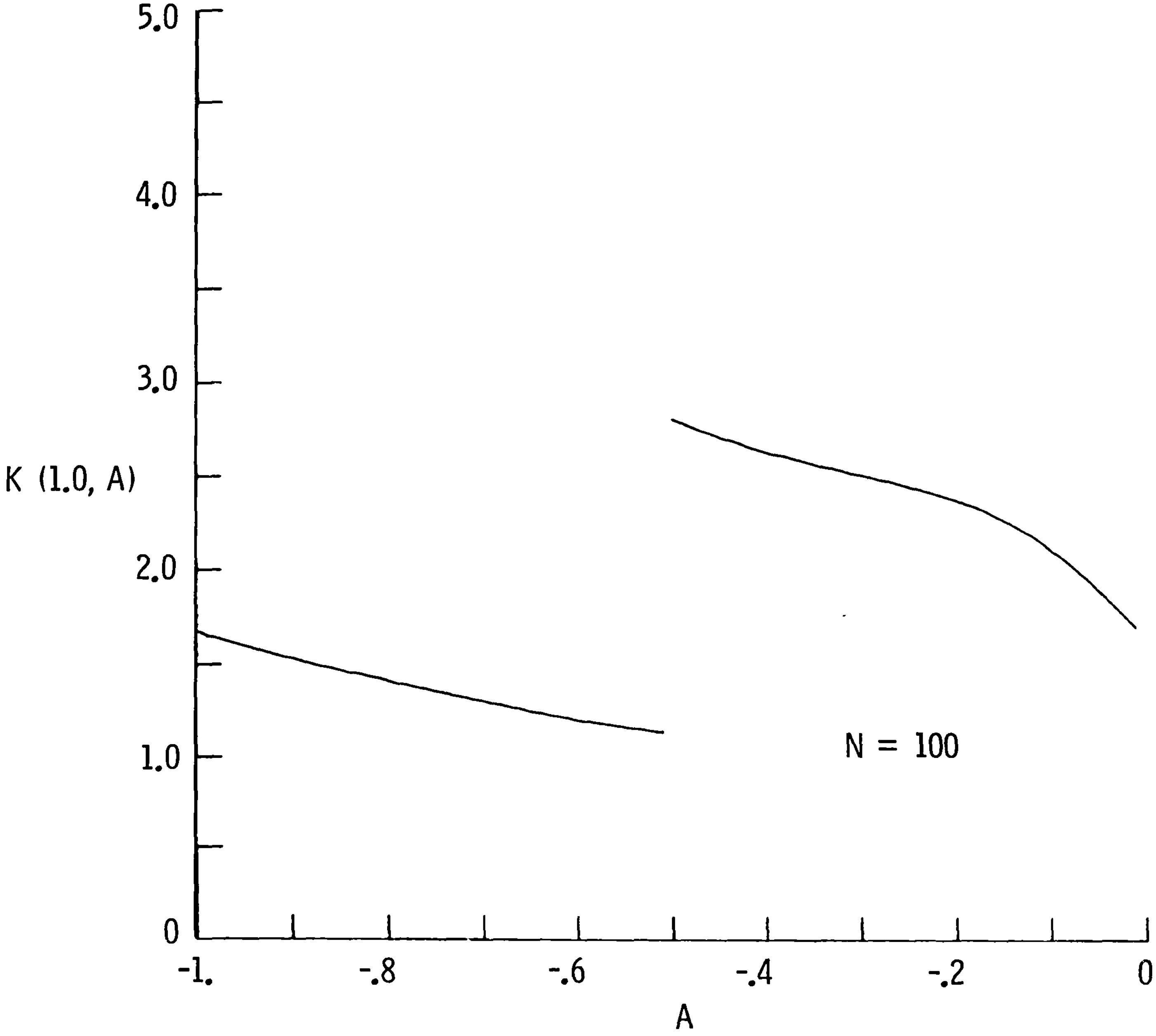

Figure 5.35

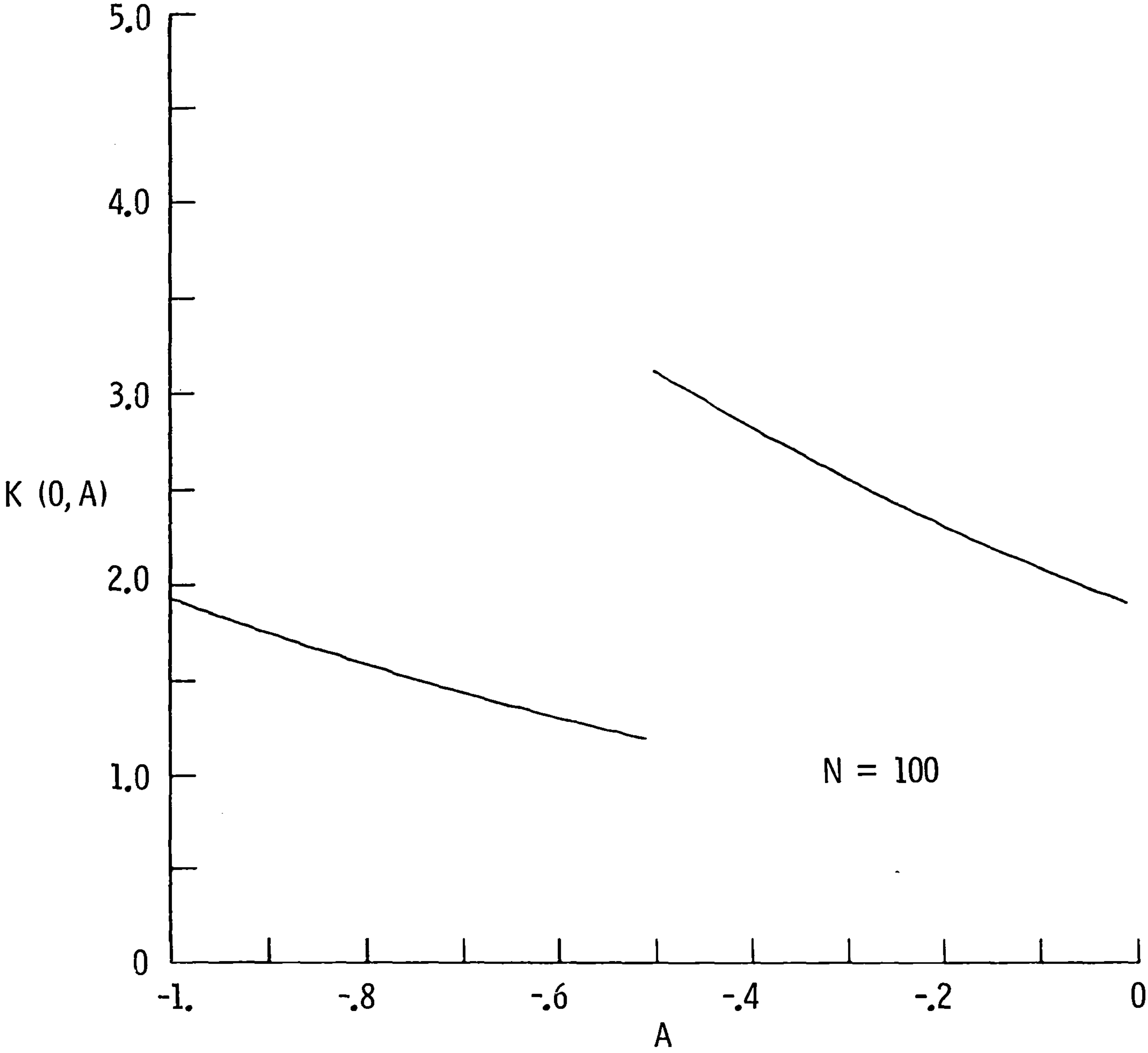

Figure 5.36

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16. Abstract

A Chandrasekhar-type factorization method is applied to the linear-quadratic optimal control problem for distributed parameter systems. An aeroelastic control problem is used as a model example to demonstrate that if computationally efficient algorithms, such as those of Chandrasekhar-type, are combined with the special structure often available to a particular problem, then an abstract approximation theory developed for distributed parameter control theory becomes a viable method of solution. A numerical scheme based on averaging approximations is applied to hereditary control problems. Numerical examples are given.

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