The realization of input-output maps using bialgebras

Robert Grossman* and Richard G. Larson†
University of Illinois at Chicago

1 Introduction

In this paper, we use the theory of bialgebras to prove a state space realization theorem for input/output maps of dynamical systems. This approach allows us to consider from a common viewpoint the classical results of Fliess Fliess [3], [4] and more recent results on realizations involving families of trees [7], [6]. The following definition is fundamental to this approach. If H be a bialgebra, we say that $p \in H^*$ is *differentially produced by the algebra R with the augmentation* ϵ if

1. there is right H-module algebra structure on R;

2. there exists $f \in R$ satisfying $p(h) = \epsilon(f \cdot h)$.

We will characterize those $p \in H^*$ which are differentially produced.

Differentially produced elements of algebras arise naturally when studying dynamical systems with inputs and outputs. For example, let R denote the field of rational functions in the variables $x_1, \ldots, x_N$ with coefficients from the field k, and let $E_1, \ldots, E_M$ denote M derivations of R. The dynamical system

$$
\begin{aligned}
\dot{x}(t) &= \sum_{\mu=1}^{M} u_\mu(t) E_\mu(x(t)), \\
x(0) &= x^0 \in \mathbf{R}^N
\end{aligned}
$$
(1)

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together with an observation function $f \in R$

$$f : \mathbf{R}^N \longrightarrow \mathbf{R} \qquad\qquad (2)$$

naturally specifies an input/output map, which is defined by sending the input functions

$$t \rightarrow u_1(t), \ldots, \quad t \rightarrow u_M(t)$$

to the output function

$$t \rightarrow f(x(t)).$$

The properties of the input/output map are captured by the formal series

$$\sum_{\text{words } \mu} c_\mu \mu,$$

where

$$c_\mu = E_{\mu_k} \cdots E_{\mu_1} f(x(0)), \qquad \mu = \mu_1 \cdots \mu_k, \text{ a word.}$$

This series is often called the generating series, while the data consisting of a dynamical system with inputs, together with an observation, are called a state space realization of the input/output map. Isadori [11] contains a detailed description of these topics, as well as extensive references.

Let H denotes the free associative algebra in the symbols $E_1, \ldots, E_M$ over the field k and let H^* denote its topological dual. H^* is isomorphic to a formal power series algebra in infinitely many variables. The point of view of this paper is to consider the formal series p as an element of the algebra H^*. If $p \in H^*$ is the formal series associated with an input/output map, then it is differentially produced. Conversely, we can ask which formal series $p \in H^*$ have the property that there is a dynamical system and an observation function which realizes it as above; that is, which p are differentially produced? We will see that both these questions are simply answered if we exploit the bialgebra structure of H.

Important work in this area has been done by Fleiss [3] and [4], Hermann and Krener [10], and Sussman [16]. Fleiss was the first to focus on the algebraic and combinatorial aspects of the problem, making important use of shuffle algebras in his study of realization theory. His work was simplified by Reutenauer [15]. In this paper, we generalize and simplify the work of Fleiss and Reutenauer, extending the context to general bialgebras. This allows us to treat combinatorial examples of differential representations which have arisen in the symbolic computation of solutions of differential equations [8] and [7].

To state the theorem we prove in Section 2, we need some definitions. Let k denote a field of characteristic 0. If V is a vector space over k, denote by V^* the set of all linear maps $V \to k$. The vector space V^* with the finite topology is a complete topological vector space (see [12] for details).

By an *algebra* over k we mean an associative algebra with identity. The algebra structure of A can be specified by the maps $A \otimes_k A \to A$ which maps $a \otimes b \in A \otimes_k A$ to $ab \in A$, and $k \to A$ which maps $1 \in k$ to $1 \in A$. The facts that multiplication is associative and that $1 \in A$ is a two-sided unit for multiplication can be expressed in terms of the commutativity of certain diagrams. An *augmentation* for the algebra A is an algebra homomorphism $A \to k$.

In a dual manner we define a *coalgebra*. A coalgebra is a vector space C over k, equipped with maps $\Delta : C \to C \otimes_k C$ and $\epsilon : C \to k$ which give a coassociative comultiplication and a counit (that is, the diagrams which are dual to those in the definition of an algebra commute). If $c \in C$ we will sometimes write the element $\Delta(c) \in C \otimes_k C$ as $\sum_{(c)} c_{(1)} \otimes c_{(2)}$, using notation introduced by Sweedler in [17]. If C is a coalgebra, then C^* is an algebra. Note that $\epsilon : C \to k$ is the multiplicative identity for the algebra C^*.

We define a *bialgebra* to be a vector space H equipped with an algebra and a coalgebra structure, so that the maps which define the coalgebra structure are algebra homomorphisms, or equivalently, the maps which define the algebra structure are coalgebra homomorphisms. In particular, the coalgebra counit ϵ is an augmentation for the algebra H. If H is a bialgebra, its *primitive* elements are defined by

$$P(H) = \{ h \in H \mid \Delta(h) = 1 \otimes h + h \otimes 1 \}.$$

It can be shown that $P(H)$ is a Lie algebra with respect to the operation $[x, y] = xy - yx$. It has been shown [14] that if the characteristic of k is 0, and H is generated as an algebra by $P(H)$ (in which case we say that H is *primitively generated*), then $H \cong U(P(H))$, where $U(L)$ denotes the universal enveloping algebra of the Lie algebra L. The Poincaré-Birkhoff-Witt Theorem (see [13]) states that if $e_1, e_2, \ldots$ is an ordered basis for L, then

$$\{ e_{i_1}^{\alpha_{i_1}} \cdots e_{i_k}^{\alpha_{i_k}} \mid i_1 < \cdots < i_k \text{ and } 0 < \alpha_{i_r} \}$$

is a basis for $U(L)$. It follows that $U(L)^*$ is a formal power series algebra. More specifically, if we denote the basis element $e_{i_1}^{\alpha_{i_1}} \cdots e_{i_k}^{\alpha_{i_k}}$ of $U(L)$ by e^α, and let $\{x_\alpha\}$ be the dual basis (in the sense of complete topological vector

spaces), then $U(L)^* \cong k[[x_1, x_2, \dots]]$. Under this isomorphism

$$x_\alpha = \frac{x^\alpha}{\alpha!},$$

where $x^\alpha = x_{i_1}^{\alpha_{i_1}} \cdots x_{i_k}^{\alpha_{i_k}}$ and $\alpha! = \alpha_{i_1}! \cdots \alpha_{i_k}!$. (Note that we can think of α as an infinite sequence of non-negative integers, all but finitely many of which are 0. Observe that $x_j^0 = 1$ and $0! = 1$.)

Let H denote an algebra with a spanning set

$$\{\xi_\alpha : \alpha \text{ a word}\}$$

indexed by all words $\alpha = \alpha_1 \cdots \alpha_i$ from an alphabet. We describe when H is a *shuffle algebra* on this spanning set. The shuffle of two words

$$\alpha = \alpha_1 \cdots \alpha_i, \quad \beta = \beta_1 \cdots \beta_j$$

is defined as follows. Let $K = \{1, 2, \dots, i+j\}$, and let

$$\lambda \; : \; \{1, \dots, i\} \longrightarrow K$$
$$\mu \; : \; \{1, \dots, j\} \longrightarrow K$$

denote two order-preserving maps such that the images $\lambda(\alpha)$ and $\mu(\beta)$ are disjoint and complementary. These data define a word $\gamma = \gamma_1 \cdots \gamma_k$ via

$$\gamma_l = \begin{cases} \alpha_{\lambda^{-1}(l)} & \text{if } l \in \operatorname{Im} \lambda \\ \beta_{\mu^{-1}(l)} & \text{if } l \in \operatorname{Im} \mu. \end{cases}$$

The *shuffle* of α and β is defined to be the set of all such γ obtained in this fashion. We can now define the shuffle algebra structure on H. The *shuffle product* of two elements ξ_α and ξ_β is defined by

$$\xi_\alpha \cdot \xi_\beta = \sum_{\gamma \in \Gamma} \xi_\gamma,$$

where Γ is the shuffle of the words α and β. The algebra H is called a shuffle algebra if the multiplication in H (with respect to the given spanning set) is given by the shuffle product.

Let H be a primitively generated bialgebra. We define a right and left H-module structure on H^* as follows: if $p \in H^*$ and $h \in H$, let $p \leftharpoonup h \in H^*$ be defined by

$$(p \leftharpoonup h)(k) = p(hk), \qquad k \in H,$$

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and let $h \rightharpoonup p \in H^*$ be defined by

$$(h \rightharpoonup p)(k) = p(kh), \qquad k \in H.$$

We say that an algebra A is a *left H-module algebra* if A is a left H-module, and

$$h \cdot (ab) = \sum_{(h)} (h_{(1)} \cdot a)(h_{(2)} \cdot b).$$

A *right H-module algebra* is defined similarly. If A is a left or right H-module algebra, we say that H *measures A to itself*. In particular, H measures H^* to itself using the actions $\rightharpoonup$ and $\leftharpoonup$ defined above. If the bialgebra H measures the algebra A to itself, then the elements of $P(H)$ act as derivations of A.

We say that $p \in H^*$ has *finite Lie rank* if $\dim P(H) \rightharpoonup p$ is finite. Recall that $p \in H^*$ is *differentially produced by the algebra R with the augmentation ϵ* if

1. there is right H-module algebra structure on R;

2. there exists $f \in R$ satisfying $p(h) = \epsilon(f \cdot h)$.

Concrete examples of differentially produced functionals on a primitively generated bialgebra (that is, of differentially produced formal power series) are given in Section 3.

Our main theorem is the following.

Theorem 1.1 *Let H be a primitively generated bialgebra over a field of characteristic 0. Let $p \in H^*$. Then the following are equivalent:*

1. p has finite Lie rank;

2. p is differentially produced by some augmented k-algebra R for which $\dim (\operatorname{Ker} \epsilon)/(\operatorname{Ker} \epsilon)^2$ is finite;

3. p is differentially produced by a subalgebra of H^* which is isomorphic to $k[[x_1, \ldots, x_N]]$, the algebra of formal power series in N variables.

We prove this theorem in Section 2. We give examples of its application in Section 3.

2 Proof of Main Theorem

We first prove that part (1) of Theorem 1.1 implies part (3). Given a fixed $p \in H^*$, we define three basic objects:

$$
\begin{aligned}
L &= \{\, h \in P(H) \mid h \rightharpoonup p = 0 \,\} \\
J &= HL \\
J^\perp &= \{\, q \in H^* \mid q(j) = 0 \text{ for all } j \in J \}.
\end{aligned}
$$

Note that L is a Lie ideal in $P(H)$, so that J is an ideal in the algebra H. Also, J is easily shown to be a coideal. Therefore $J^\perp \cong (H/J)^*$ is a subalgebra of H^*. We will show that $J^\perp$ is isomorphic to a formal power series algebra, and will construct derivations of this ring which will be used to realize the input/output map defined by p.

Lemma 2.1 *If* $\dim P(H) \rightharpoonup p = N$, *then* $J^\perp$ *is a subalgebra of* H^* *satisfying*

$$
J^\perp \cong k[[x_1, \ldots, x_N]].
$$

PROOF. The Lie ideal L has finite codimension N. Choose a basis $\{e_1, e_2, \ldots\}$ of $P(H)$ such that $\{e_{N+1}, e_{N+2}, \ldots\}$ is a basis of L. Note that if $\bar{e}_i$ is the image of e_i under the quotient map $P(H) \to P(H)/L$, then the image of $\{\bar{e}_1, \ldots, \bar{e}_N\}$ is a basis for $P(H)/L$.

By the Poincaré-Birkhoff-Witt Theorem, H has a basis of the form

$$
\{\, e_{i_1}^{\alpha_{i_1}} \cdots e_{i_k}^{\alpha_{i_k}} \mid i_1 < \cdots < i_k \text{ and } 0 < \alpha_{i_r} \,\}.
$$

Since L is a Lie ideal, $J = HL$ is an ideal in the algebra H, and has a basis of the form

$$
e_{i_1}^{\alpha_{i_1}} \cdots e_{i_k}^{\alpha_{i_k}}
$$

with at least one $i_r > N$. Specifically $U(P(H)/L) \cong H/J$, and

$$
\{\, \bar{e}_1^{\alpha_1} \cdots \bar{e}_N^{\alpha_N} \mid \alpha_1, \ldots, \alpha_N \geq 0 \,\}
$$

is a basis for $U(P(H)/L)$. It follows that the elements of the form

$$
x_\alpha = \frac{x^\alpha}{\alpha!} = \frac{x_{i_1}^{\alpha_{i_1}} \cdots x_{i_k}^{\alpha_{i_k}}}{\alpha_{i_1}! \cdots \alpha_{i_k}!}
$$

with all $1 \leq x_r \leq N$ are in $J^\perp \subseteq H^*$. Indeed, $J^\perp$ consists precisely of the completion in the finite topology of the span of such elements. In other words,

$$J^\perp \cong k[[x_1, \ldots, x_N]],$$

completing the proof. ∎

We will use the following notation and facts from the proof of Lemma 2.1: Suppose that $\{e_1, \ldots\}$ is a basis for $P(H)$ such that $\{e_{N+1}, \ldots\}$ is a basis for L. Let $\{e^\alpha\}$ be the corresponding Poincaré-Birkhoff-Witt basis. Denote $J^\perp$ by R. Then $R \cong k[[x_1, \ldots, x_N]]$, and $x_1^{\alpha_1} \cdots x_N^{\alpha_N}/\alpha_1! \cdots \alpha_N!$ equals the element of the dual (topological) basis of H^* to the Poincaré-Birkhoff-Witt basis $\{e^\alpha\}$ of H, corresponding to the basis element $e_1^{\alpha_1} \cdots e_N^{\alpha_N}$.

We now collect some properties of the ring of formal power series R which will be necessary for the proof of the theorem.

Lemma 2.2 *Assume $p \in H^*$ has finite Lie rank, and let $R \subseteq H^*$, $e_\alpha \in H$, and $x^\alpha \in R$ be as above. Define*

$$f = \sum_{\substack{\alpha = (\alpha_1, \ldots, \alpha_N) \\ \alpha \geq 0}} c_\alpha x^\alpha \in R,$$

where $c_\alpha = \dfrac{p(e^\alpha)}{\alpha!}$. Then

1. *H measures R to itself via $\rightharpoonup$;*
2. *$p(h) = \epsilon(f \rightharpoonup h)$ for all $h \in H$.*

PROOF. We begin with the proof of part (1). Since H measures H^* to itself and $R \subseteq H^*$, we need show only that $R \rightharpoonup H \subseteq R$. Take $r \in R$, $h \in H$ and $j \in J$. We have $(r \rightharpoonup h)(j) = r(hj)$. Since J is an ideal, $hj \in J$, so $r(hj) = 0$, so $r \rightharpoonup h \in J^\perp = R$. This proves part (1).

We now prove part (2). Let $e^\alpha = e_{i_1}^{\alpha_{i_1}} \cdots e_{i_k}^{\alpha_{i_k}}$ be a Poincaré-Birkhoff-Witt basis element of H. Since $e^\alpha \in J$ unless $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, N\}$, $p(e^\alpha) = 0$ unless $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, N\}$. Also $\epsilon(f \rightharpoonup e^\alpha) = f \rightharpoonup e^\alpha(1) = f(e^\alpha 1) = f(e^\alpha) = 0$ unless $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, N\}$. Now suppose $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, N\}$. We have in this case also that $p(e^\alpha) = \alpha! c_\alpha = f(e^\alpha) = f \rightharpoonup e^\alpha(1) = \epsilon(f \rightharpoonup e^\alpha)$. Since $\{e^\alpha\}$ is a basis for H, this completes the proof of part (2) of the lemma. ∎

Corollary 2.1 *Under the assumptions of Lemma 2.2, $f = p$.*

Lemmas 2.1 and 2.2 yield that part (1) implies part (3) in Theorem 1.1. It is immediate that part (3) implies part (2).

We now complete the proof of Theorem 1.1 by proving that part (2) implies part (1).

Let $x_1, \ldots, x_N$ be chosen so that $\{\bar{x}_1, \ldots, \bar{x}_N\}$ is a basis for $(\operatorname{Ker}\epsilon)/(\operatorname{Ker}\epsilon)^2$. If $f \in R$ and $h \in H$, then

$$f \cdot h = q_0(h)1 + \sum_{i=1}^{N} q_i(h)x_i + g(h),$$

where $q_i \in H^*$ and $g(h) \in (\operatorname{Ker}\epsilon)^2$. Let $l \in P(H)$. Since H measures R to itself and $\Delta(l) = 1 \otimes l + l \otimes 1$, the map $f \mapsto f \cdot l$ is a derivation of R. Now let $f \in R$ be the element such that

$$p(h) = \epsilon(f \cdot h).$$

Then

$$
\begin{aligned}
f \cdot hl &= (f \cdot h) \cdot l \\
&= q_0(h)1 \cdot l + \sum_{i=1}^{N} q_i(h)x_i \cdot l + g(h) \cdot l.
\end{aligned}
$$

Since the map $f \mapsto f \cdot l$ is a derivation, $1 \cdot l = 0$; since $g(h) \in (\operatorname{Ker}\epsilon)^2$, $g(h) \cdot l \in \operatorname{Ker}\epsilon$. It follows that

$$
\begin{aligned}
l \rightharpoonup p(h) &= p(hl) \\
&= \epsilon(f \cdot hl) \\
&= \sum_{i=1}^{N} q_i(h)\epsilon(x_i \cdot l).
\end{aligned}
$$

Therefore $P(H) \rightharpoonup p \subseteq \sum_{i=1}^{N} kq_i$, so p has finite Lie rank. This completes the proof of Theorem 1.1

3 Examples

In this section, we discuss two examples of applications of Theorem 1.1. The first example is obtained by setting the bialgebra H to be the free associative algebra over the field k in the symbols $E_1, \ldots, E_M$. This example motivated the theorem and is the basic setting in the control theory literature (see [3],

[4] and [15]). The second example is obtained by setting the bialgebra H to be families of trees with the appropriate multiplication and comultiplication. This example arises when studying algorithms for the symbolic computation of higher order derivations generated by derivations $E_1, \ldots, E_M$; see [8] and [9]. There is a natural homomorphism between these two Hopf algebras which is described in [8].

Example 1. We begin by giving a description of the setting for this example. Let R denote the field of rational functions in the variables $x_1, \ldots, x_N$ with coefficients from the field k, and let $E_1, \ldots, E_M$ denote M derivations of R. The algebras in this example are associated with a pair consisting of the dynamical system (1) and the observation function (2) introduced in Section 1. We assume that the controls

$$t \to u_1(t), \ldots, \quad t \to u_M(t)$$

in (1) are continuous and bounded almost everywhere.

Integrating the initial value problem (1) gives

$$f(x(t)) = f(x(0)) + \sum_{\mu=1}^{M} \int_0^t u_\mu(\tau_1) E_\mu(f(x(\tau))) \, d\tau.$$

Integrating again gives

$$
\begin{aligned}
f(x(t)) \;=\; & f(x(0)) + \sum_{\mu_1=1}^{M} E_{\mu_1}(f(x(0))) \int_0^t u_{\mu_1}(\tau) \, d\tau_1 \\
& + \sum_{\mu_1,\mu_2=1}^{M} \int_0^t \int_0^{\tau_1} u_{\mu_1}(\tau_1) u_{\mu_2}(\tau_2) E_{\mu_1} E_{\mu_2}(f(x(\tau_2))) \, d\tau_2 d\tau_1.
\end{aligned}
$$

Continuing this process yields

$$
\begin{aligned}
f(x(t)) \;=\; & f(x(0)) + \sum_{\mu_1=1}^{M} \int_0^t u_{\mu_1}(\tau) \, d\tau_1 \, E_{\mu_1}(f(x(0))) \\
& + \ldots + E_{\mu_k} \cdots E_{\mu_1}(f(x(0))) \cdot \\
& \qquad \sum_{\mu_1,\ldots,\mu_k=1}^{M} \int_0^t \cdots \int_0^{\tau_{k-1}} u_{\mu_1}(\tau_1) \cdots u_{\mu_k}(\tau_k) \, d\tau_k \cdots d\tau_1 \\
& + \text{remainder},
\end{aligned}
$$

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where the remainder is of the form

$$\sum_{\mu_1,\ldots,\mu_{k+1}=1}^{M} \int_0^t \cdots \int_0^{\tau_k} u_{\mu_1}(\tau_1) \cdots u_{\mu_{k+1}}(\tau_{k+1})$$

$$\cdot E_{\mu_{k+1}} \cdots E_{\mu_1}(f(x(\tau_{k+1}))) \, d\tau_{k+1} \cdots d\tau_1.$$

Let $\mu = \mu_1 \cdots \mu_k$ denote a word of length k built from the alphabet $\{1, \ldots, M\}$. The process above defines a formal series

$$\bar{p} = \sum_{\text{words } \mu} c_\mu \bar{\xi}_\mu, \qquad (3)$$

where

$$c_\mu = E_{\mu_k} \cdots E_{\mu_1} f(x(0)) \in k, \qquad (4)$$

and

$$\bar{\xi}_\mu(t) = \begin{cases} \displaystyle\int_0^t u_{\mu_1}(\tau) \, d\tau & \text{if} \quad \mu = \mu_1 \\ \displaystyle\int_0^t u_{\mu_1}(\tau) \bar{\xi}_{\mu_2 \cdots \mu_k}(\tau) d\tau & \text{if} \quad \mu = \mu_1 \ldots \mu_k. \end{cases} \qquad (5)$$

It is a theorem of Chen [1] that functions of the form (5) form a shuffle algebra, that is, that

$$\bar{\xi}_\lambda(t) \cdot \bar{\xi}_\mu(t) = \sum_\nu \bar{\xi}_\nu(t),$$

where the sum is over words ν that are in the shuffle of λ and μ.

Let $H = k<E_1,\ldots,E_M>$ denotes the free associative algebra in the symbols $E_1,\ldots,E_M$ over the field k. Recall that H is a bialgebra. The coproduct and counit are defined by letting

$$\begin{aligned} \Delta(E_i) &= 1 \otimes E_i + E_i \otimes 1, \\ \epsilon(E_i) &= 0, \end{aligned}$$

for $i = 1,\ldots, M$, and then extending to all of $k<E_1,\ldots,E_M>$ by requiring that Δ and ϵ be algebra homomorphisms. The bialgebra H is cocommutative, but not commutative. The algebra of formal series in the $\bar{\xi}_\mu(t)$ is a quotient of the algebra H^*.

The papers of Fliess [3], [4], Reutenauer [15], and Crouch and Lamnabhi-Lagarrigue [2] all view the formal series $\bar{p}$ above as an element of the shuffle algebra of formal power series in the noncommuting variables $E_1,\ldots,E_M$. It is easy to relate that point of view to the point of view taken here. The bialgebra H has basis consisting of all words E_μ in the generators $E_1,\ldots,E_M$,

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including the empty word 1. Let ξ_ν denote the elements in the dual H^* of H which are formally dual to the E_μ, that is,

$$\xi_\nu(E_\mu) = \delta_{\mu,\nu},$$

where $\delta_{\mu,\nu}$ is the Kronecker delta. Then the ξ_ν can be viewed as a topological basis for the formal non-commutative power series ring over k generated by $E_1, \ldots, E_M$. As we observed in Section 1, the bialgebra H^* is a commutative algebra with respect to the shuffle product on the ξ_μ. Fixing a control system (1) and an observation function (2) determines an element of H^*

$$p = \sum_{\text{words } \mu} c_\mu \xi_\mu,$$

where the c_μ are given by Equation (4). The element $\bar{p}$ given by Equation (3) can be viewed as an element of a quotient algebra of H^*. Theorem 1.1 applied to this example gives the classical theorem of Fliess [3], [4].

Example 2. We follow [8] and [9] for this example. By a tree we mean a rooted finite tree [18]. If $\{E_1, \ldots, E_M\}$ is a set of symbols, we will say a tree is *labeled with* $\{E_1, \ldots, E_M\}$ if every node of the tree other than the root has an element of $\{E_1, \ldots, E_M\}$ assigned to it. We denote the set of all trees labeled with $\{E_1, \ldots, E_M\}$ by $\mathcal{LT}(E_1, \ldots, E_M)$. Let $k\{\mathcal{LT}(E_1, \ldots, E_M)\}$ denote the vector space over k with basis $\mathcal{LT}(E_1, \ldots, E_M)$. We show that this vector space is a graded connected Hopf algebra.

We define the multiplication in $k\{\mathcal{LT}(E_1, \ldots, E_M)\}$ as follows. Since the set of labeled trees form a basis for $k\{\mathcal{LT}(E_1, \ldots, E_M)\}$, it is sufficient to describe the product of two labeled trees. Suppose t_1 and t_2 are two labeled trees. Let $s_1, \ldots, s_r$ be the children of the root of t_1. If t_2 has $n+1$ nodes (counting the root), there are $(n+1)^r$ ways to attach the r subtrees of t_1 which have $s_1, \ldots, s_r$ as roots to the labeled tree t_2 by making each s_i the child of some node of t_2, keeping the original labels. The product $t_1 t_2$ is defined to be the sum of these $(n+1)^r$ labeled trees. It can be shown that this product is associative, and that the tree consisting only of the root is a multiplicative identity (see [5] or [6] for details).

We define the comultiplication Δ on $k\{\mathcal{LT}(E_1, \ldots, E_M)\}$ as follows. Let t be a labeled tree, and let $s_1, \ldots, s_r$ be the children of the root of t. If P is a subset of $C_t = \{s_1, \ldots, s_r\}$, let t_P be the labeled tree formed by making the elements of P the children of a new root, keeping the original labels. Define $\Delta(t) = \sum_{P \subseteq C_t} t_P \otimes t_{C_t \setminus P}$, where $X \setminus Y$ denotes the set-theoretic relative

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complement of Y in X. Define the augmentation $\epsilon(t)$ of the bialgebra to be 1 if t has only one node (its root), and 0 otherwise. We define a grading on $k\{\mathcal{LT}(E_1, \ldots, E_M)\}$ by letting $k\{\mathcal{LT}(E_1, \ldots, E_M)\}_n$ be the subspace of $k\{\mathcal{LT}(E_1, \ldots, E_M)\}$ spanned by the trees with $n+1$ nodes. The following theorems are proved in [6].

Theorem 3.1 $H = k\{\mathcal{LT}(E_1, \ldots, E_M)\}$ *is a cocommutative graded connected bialgebra.*

Theorem 3.2 *The set of labeled trees t whose root has exactly one child is a basis for the primitives $P(H)$ of $H = k\{\mathcal{LT}(E_1, \ldots, E_M)\}$.*

Let R denote the field of rational functions in the variables $x_1, \ldots, x_N$ with coefficients from the field k, and let $E_1, \ldots, E_M$ denote M derivations of R of the form

$$E_\gamma = \sum_{\mu=1}^N b_\gamma^\mu \frac{\partial}{\partial x_\mu},$$

where $b_\gamma^\mu \in R$. We now define an H-module algebra on R. The action of H on R is given by the map $\psi : H \to \text{End}_k R$, which is defined as follows.

1. Given a labeled tree t with $m+1$ nodes, assign the root the number 0 and assign the remaining nodes the numbers $1, \ldots, m$. We identify the node with the number assigned to it. To the node k asociate the summation index μ_k. Denote $(\mu_1, \ldots, \mu_m)$ by μ.

2. For the labeled tree t, let k be a node of t, labeled with E_{γ_k} if $k \neq 0$, and let $l, \ldots, l'$ be the children of k. Define

$$c(k; \mu) = \begin{cases} \dfrac{\partial}{\partial x_{\mu_l}} \cdots \dfrac{\partial}{\partial x_{\mu_{l'}}} b_{\gamma_k}^{\mu_k}(x) & \text{if } k \neq 0 \text{ is not the root;} \\[2ex] \dfrac{\partial}{\partial x_{\mu_l}} \cdots \dfrac{\partial}{\partial x_{\mu_{l'}}} & \text{if } k = 0 \text{ is the root.} \end{cases}$$

Note that if $k \neq 0$, then $c(k; \mu) \in R$.

3. Define

$$\psi(t) = \sum_{\mu_1, \ldots, \mu_m = 1}^N c(m; \mu) \cdots c(1; \mu) c(0; \mu).$$

4. Extend ψ to all of H by linearity.

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It is straight-forward to check that this action of H on R makes R into a H-module algebra.

An element $p \in H^*$ can be thought of as an infinite series whose terms are indexed by labeled trees rather than by words, as well as an element of a power series algebra. Theorem 1.1 gives necessary and sufficient conditions for p to be differentially produced in this case.

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