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Review of Probabilistic Analysis of
Dynamic Response of Systems
with Random Parameters $/N-39$
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I. Introduction

The objective of this report is to review the various methods that have been studied in the past to allow probabilistic analysis of dynamic response for systems with random parameters. In general, the mechanical parameters (i.e., spring, damping, joint parameters, dead zones, etc.) may not be known exactly. If, for example, the variations about the nominal values are very small, then the dynamic response would be adequately obtained deterministically. However, for space structures which require precise pointing, it appears that the variations or uncertainties about the nominal values of the structural details and of the environmental conditions may be too large to be considered as negligible.

Thus, these uncertainties must be accounted for on some rational basis which we shall assume to be defined in terms of probability distributions about their nominal values. The quantities of concern for describing the response of the structure includes displacements and velocities, as well as the distributions of natural frequencies. The exact statistical characterization of the response would yield joint probability distributions for the response variables. Since the random quantities will appear as coefficients, determining the exact distributions will be difficult at best. Thus, certain approximations will have to be made. There are a number of techniques that we shall discuss that are available even in the non-linear case.

In the general case, the n-mass linear structural system possesses the dynamical description through the differential equation

$$M\ddot{y} + C\dot{y} + Ky = f(t) \qquad (1.1)$$

where M, C, K are the mass, damping and stiffness matrices, and f(t) denotes the external excitation on the structure.

We shall define the vector equations (1.1) through the vector

$$x = \begin{pmatrix} y \\ \dot{y} \end{pmatrix} \qquad (1.2)$$

which represents the system (1.1) as

$$\dot{x} = Ax + Bf(t), \quad x_0 = \begin{pmatrix} y_0 \\ \dot{y}_0 \end{pmatrix} \qquad (1.3)$$

where

$$A = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{pmatrix} , \quad B = \begin{pmatrix} 0 \\ M^{-1} \end{pmatrix} \qquad (1.4)$$

The solution of (1.3) for $x_0(0) = x_0$ is

$$x(t) = e^{-At}x_0 + \int_0^t e^{-A(t-\tau)}Bf(\tau)d\tau \quad . \qquad (1.5)$$

We assume that the matrix B, determined by the mass constants, is deterministic. Thus, the random quantities appear in matrix A. We shall denote the random variable in A, as $X_1, \ldots, X_m$. Further, the external excitations $f(t)$ may or may not be random.

The most important topic for engineering systems is how uncertain parameter values influence the accuracy of system response prediction. It often suffices to know how these uncertainities influence the accuracy in estimating the values of the national frequencies and their corresponding normal modes of motion in a conservative system $(C = 0)$.

Since linear system response prediction depends upon frequency response or impulsive admittance, our interest will center on natural frequencies, normal modes, frequency response, as well as impulse response. The methods of techniques that we will describe in order to pursue the various subjects are: (1) Liouville's equation; (2) perturbation methods; (3) mean square approximate

systems; and (4) non-linear systems, with approximation by linear systems.

II. Liouville Equation

In this section we derive the method based upon the Liouville equation for the time evolution of the joint probability distribution function of the state space $(2n \times 1)$ column vector x and the system parameters.

The use of the Liouville equation in mechanics and statistical mechanics is of long standing and goes back to Maxwell (see for example [1,2,3]). These references do not consider random system parameters, and average quantities under equilibrium conditions is of main interest. While not of direct interest to us, it is possible to adapt these early methods to our needs. We derive the needed form of the Liouville equation following a procedure suggested by Kozin [4] for systems with random parameters and random or deterministic initial values.

We are interested in the linear equations of motion in the form given by (1.3) with f= 0:

$$\dot{x} = Ax \ , \tag{2.1}$$

where x is the $(2n \times 1)$ colwumn vector whose transpose x^T has the form $x^T = \left\{ x_1, \ldots, x_n; \dot{x}_1, \ldots, \dot{x}_n \right\}$ and A is the $(2n \times 2n)$ matrix given by the first of (1.4) The vector x is the state space form for representing the system response; the components of x will be denoted by $x_k(t)$, k=1, . . . , 2n. The random variables in A are denoted by $X_1, \ldots, X_m$. However, since the Liouville approach applies to general nonlinear as well as linear equations, we consider the general system

$$\dot{x}_k = g_k(x_1, \ldots, x_{2n}; X_1, \ldots, X_m; t) \ , \quad k=1, \ldots, 2n \ . \tag{2.2}$$

Let $p(x_1, \ldots, x_{2n}; X_1, \ldots, X_n; t)$ be the joint probability distribution of the random quantities $(x_1, \ldots, x_{2n}; X_1, \ldots, X_n)$. We define the characteristic function ϕ as

$$\phi = E \left[\exp i \left(\sum_{k=1}^{2n} \theta_k x_k(t) + \sum_{j=1}^{m} \phi_j X_j \right) \right] \ , \quad i = \sqrt{-1} \ . \qquad (2.3)$$

The differentiation of (2.3) with respect to time gives

$$\frac{\partial \phi}{\partial t} = E \left[i \sum_{k=1}^{2n} \theta_k \dot{x}_k(t) \, \exp i \left(\sum_{k=1}^{2n} \theta_k x_k(t) + \sum_{j=1}^{m} \phi_j X_j \right) \right] \ . \qquad (2.4)$$

The use of (2.2) in (2.4) yields

$$\frac{\partial \phi}{\partial t} = i \sum_{k=1}^{2n} \theta_k E \left[g_k \exp i \left(\sum_{k=1}^{2n} \theta_k x_k(t) + \sum_{j=1}^{m} \phi_j X_j \right) \right] \qquad (2.5)$$

Since (2.3) is essentially the Fourier transform of the joint density function $p(x_1,...,x_{2n}; X_1,...X_m; t)$, the inverse Fourier transform of (2.5) produces

$$\frac{\partial p}{\partial t} = - \sum_{j=1}^{2n} \frac{\partial (g_j p)}{\partial x_j} \ . \qquad (2.6)$$

The solution of (2.6) for p is given by a suitable function of the independent integrals of the Lagrangian system

$$\frac{dt}{1} = \frac{-dp}{p \left(\dfrac{\partial g_1}{\partial x_1} + ... + \dfrac{\partial g_{2n}}{\partial x_{2n}} \right)} = \frac{dx_1}{g_1} = \cdots = \frac{dx_{2n}}{g_{2n}} \ . \qquad (2.7)$$

Let $u_1, \ldots, u_{2n}$ be 2n-independent integrals of (2.7). Then we know that the general solution of (2.6) is

$$p \left(x_1, \ldots, x_{2n}; X_1, \ldots, X_m; t \right) = h \left(u_1, \ldots, u_{2n}; X_1, \ldots, X_m; t \right) \qquad (2.8)$$

where h is an arbitrary function whose form is determined by the initial conditions on x.

In particular, consider the case where $X_1, \ldots, X_m$ are explicit random mechanical parameters which are independent of the integrals. Then (2.8) can be written as

$$p(x_1, \ldots, x_{2n}; X_1, \ldots, X_m; t) = h_1(u_1, \ldots, u_{2n}; t) h_2(X_1, \ldots, X_m) \ , \qquad (2.9)$$

where h_2 is the joint density function of the parameters, and h_1 is the conditional density of the randomness of the initial conditions. To illustrate the

form, consider as an example the simple linear system defined as

$$\left.\begin{array}{l} \ddot{x}(t) + \omega^2 x(t) = 0 \\ x(t_o) = x_{10} \ , \ \dot{x}(t_o) = x_{20} \end{array}\right\} \tag{2.10}$$

In this case, we would have (x_{10}, x_{20}) random initial values, and ω, a random parameter. The Liouville equation is simply written as

$$\frac{\partial p}{\partial t} = -x_2 \frac{\partial p}{\partial x_1} + \omega^2 x_1 \frac{\partial p}{\partial x_2} \tag{2.11}$$

where (2.7) is simply

$$\frac{dt}{1} = \frac{dx_1}{x_2} = -\frac{dx_2}{\omega^2 x_1} = \frac{dp}{0} \tag{2.12}$$

In this case, we would easily find

$$p(x_1, x_2; \omega; t) = h_1\big(u_1(x_{10}, x_{20}, t, t_o; \omega), \ u_2(x_{10}, x_{20}, t, t_o; \omega)\big) h_2(\omega)$$

$$= h_1 \bigg[x_{10} \cos \omega(t-t_o) + \frac{x_{20}}{\omega} \sin \omega(t-t_o) \ ,$$

$$-\omega x_{10} \sin \omega(t-t_o) + x_{20} \cos \omega(t-t_o) \bigg] h_2(\omega) \ . \tag{2.13}$$

Upon utilizing the fact that the initial values can be defined as

$$x_{10} = x_1 \cos \omega(t_o - t) + \frac{x_2}{\omega} \sin \omega(t_o - t) \equiv u_1(x_1, x_2, t_o, t, \omega)$$

$$\tag{2.14}$$

$$x_{20} = -\omega x_1 \sin \omega(t_o - t) + x_2 \cos \omega(t_o - t) \equiv u_2(x_1, x_2, t_o, t, \omega) \ ,$$

the final probability form for the simple oscillator (2.10) becomes

$$p(x_1, x_2; \omega; t) = h_1\big(u_1(x_1, x_2, t_o, t; \omega), \ u_2(x_1, x_2, t_o, t; \omega)\big) h_2(\omega) \ . \tag{2.15}$$

For the higher order system, the exact form of p in (2.8) would be obtained as in (2.15).

We note that the initial conditions may be deterministic so that h_1 is a product of impulses at the origin and at unity

$$h_1(u_1, \ldots, u_{2n}; t) = \delta(u_1)\ldots\delta(u_n)\delta(u_{n+1}-1)\ldots\delta(u_{2n}) \quad , \qquad (2.16)$$

where $\delta(.)$ is the delta function. Thus (2.15) would become

$$p(x_1, \ldots, x_{2n}; X_1, \ldots, X_m; t) = \delta(u_1)\ldots\delta(u_n)\delta(u_{n+1}-1)\ldots\delta(u_{2n})h_2(X_1, \ldots, X_m) \quad . \qquad (2.17)$$

Let us illustrate this process by the simplification of the example (2.10)-(2.15). Consider again the undamped one degree of freedom linear oscillator. Let ω^2 be a random parametric value. For $t_0 = 0$,

$$u_1 = x_1\cos\omega t - \frac{x_2}{\omega}\sin\omega t \quad ,$$

$$u_2 = \omega x_1\sin\omega t + x_2\cos\omega t \quad . \qquad (2.18)$$

Assume that ω^2 has a discrete distribution given by

$$h_2(\omega^2) = \sum_1^m P_i\delta(\omega^2 - \omega_i^2) \quad . \qquad (2.19)$$

For $x_1 = 0$, $x_2 = 1$ at $t = 0$, (2.17) becomes

$$p(x_1, x_2, \omega^2; t) = \delta(u_1)\delta(u_2 - 1)\sum_1^m P_i\delta(\omega^2 - \omega_i^2)$$

$$= \delta\left(x_1\cos\omega t - \frac{x_2}{\omega}\sin\omega t\right) \delta\left(\omega x_1\sin\omega t + x_2\cos\omega t - 1\right)\sum_1^m P_i\delta(\omega^2 - \omega_i^2) \qquad (2.20)$$

Let us determine the mean of x_1 to illustrate a possible use for (2.20); we have

$$E\{x_1\} = \int dx_1\int dx_2\int d\omega^2 \left[x_1 p(x_1, x_2, \omega^2; t\right] \quad . \qquad (2.21)$$

Straight integration of (2.21) (see [7]) yields from (2.20)

$$E\{x_1\} = \sum_1^m P_i \frac{\sin\omega_i t}{\omega_i} \quad . \qquad (2.22)$$

Other illustrations, including damping, are given in [4,5]. We are frequently concerned with the moments of x. Let us show how (2.6) can be employed to obtain them.

To keep the details simple, consider the linear damped one degree of freedom system with equation of motion,

$$\dot{x}_1 = x_2 \ (= g_1)$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{c}{m}x_2 \ (= g_2) \ .$$

(2.23)

Equation (2.6) now takes the form

$$\frac{\partial p}{\partial t} + \frac{\partial(g_1 p)}{\partial x_1} + \frac{\partial(g_2 p)}{\partial x_2} = 0 \ .$$

(2.24)

Assume the m, k, c are independent of the initial vector, with $h_2(m,k,c)$ the probability density function of these parameters, and write

$$p(x_1,x_2 t;m,k,c) = h_1\big(u_1(x_1,x_2,t,m,k,c), \ u_2(x_1,x_2,t,m,k,c)\big) \ h_2(m,k,c)$$

(2.25)

In this case, we could simply rewrite the function h_1 as

$$h_1\big(u_1(x_1,x_2,t,m,k,c)\big) = p_1(x_1,x_2,t \,|\, m,k,c)$$

(2.26)

since the parameters (m,k,c) are conditional for h_1. Upon inserting (2.26) into (2.25) and then into (2.24), we obtain

$$\frac{\partial p_1}{\partial t} + \frac{\partial(g_1 p_1)}{\partial x_1} + \frac{\partial(g_2 p_1)}{\partial x_2} = 0 \ .$$

(2.27)

where $h_2(m,k,c)$ has been factored out. Let us evaluate the conditional expectations

$$E\{x_1 \,|\, m,k,c\} = m_{1,0}(t) = \int \int x_1 p_1 dx_1 dx_2$$

(2.28)

$$E\{x_2 \,|\, m,k,c\} = m_{0,1}(t) = \int \int x_2 p_1 dx_1 dx_2$$

Differentiation (partial) of these equations with respect to time produces

$$\dot{m}_{1,0} = \int \int x_1 \frac{\partial p_1}{\partial t} \ dx_1 dx_2$$

(2.29)

$$\dot{m}_{0,1} = \int \int x_2 \frac{\partial p_1}{\partial t} \ dx_1 dx_2$$

But, from (2.23),

$$\frac{\partial p_1}{\partial t} = -\frac{\partial(g_1 p_1)}{\partial x_1} - \frac{\partial(g_2 p_1)}{\partial x_2} \ .$$

(2.30)

The substitution of (2.30) into (2.29), the employment of (2.23) and simple

integration by parts of the resulting terms on the right-hand side of (2.29) finally yields

$$\dot{m}_{1,0} = m_{0,1} \quad ,$$

$$(2.31)$$

$$\dot{m}_{0,1} = - \frac{k}{m} m_{10} - \frac{c}{m} m_{0,1} \quad .$$

The same procedure will produce the equations for the conditional moments $E\{x_1^2 | m,k,c\}$, $E\{x_2^2 | m,k,c\}$, etc. We note that for the first conditional moments we could have taken the conditional expectation of (2.23) to produce (2.31); however, this procedure only applies to the first moments.

We integrate the moment equations (2.31) to obtain the conditional moments as a function of time. On multiplying these moments by $h_2(m,k,c)$ and integrating over m,k, and c, we obtain the moments of x_1 and x_2.

It is clear from the above discussion that the Liouville equation will provide the exact solution for the joint probability density function $p(x_1, \ldots, x_{2n};$ $X_1, \ldots, X_m; t)$ in the absence of external forces provided the integrals $u_1, \ldots, u_{2n}$ can be obtained. Further, it provides a straightforward method for determining the moments of x from which means and variances of x can be obtained.

The Liouville equation applies when there are no external forces. We are interested in the case when external forces are present, of course. Let us see what can be done along these lines.

The Fokker-Planck equation is the natural extension of the Liouville equation (see [6,7]). We confine our attention to the case in which the external force vector f can be obtained by passing gaussian white noise through a stable linear damped system. We have as equations of motion, conditional on $M = m$, $K = k$, and $C = c$,

$$dx_1 = x_2 dt \ ,$$

$$dx_2 = - \frac{k}{m} x_1 dt - \frac{c}{m} x_2 dt + \frac{x_3}{m} dt \ , \qquad (2.32)$$

$$dx_3 = - \beta x_3 dt + dB \ , \quad x_3 = 0 \ \text{at} \ t=0$$

where we have employed the differential notation in this case, set $f=x_3$, and were dB is the Brownian motion increment with

$$E\{dB\} = 0 \quad , \quad E\{(dB)^2\} = \sigma^2 dt \ . \qquad (2.33)$$

The last equation of (2.32) represents the fact that the excitation is obtained by passing a gaussian white noise through a linear first order stable filter. We notice that for the Ito system (2.32) $x^T = \{x_1, x_2, x_3\}$ is a vector Markoff process that generates a Fokker-Planck equation.

It can be shown that in this case the Fokker-Planck equation for the conditional probability density function p_1 is

$$\frac{\partial p_1}{\partial t} = - \frac{\partial}{\partial x_1}(x_2 p_1) - \frac{\partial}{\partial x_2} \left\{ \left[- \frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}x_3 \right] p_1 \right\} - \frac{\partial}{\partial x_3} (- \beta x_3 p_1)$$
$$+ \frac{\sigma^2}{2} \frac{\partial^2 p_1}{\partial x_3^2} \ .$$

$$(2.34)$$

We observe that all but the last term on the right are the same as would have occurred in the Liouville equation in the absence of f. Let the conditional moments be

$$m_{k_1, k_2, k_3} = E\left\{ x_1^{k_1} \ x_2^{k_2} \ x_3^{k_3} \right\} \qquad (2.35)$$

$$= \int \int \int x_1^{k_1} \ x_2^{k_2} \ x_3^{k_3} \ p_1(x_1, x_2, x_3) \ dx_1 dx_2 dx_3$$

Then, proceeding as in the development of (2.31), we find

$$\dot{m}_{1,0,0} = m_{0,1,0}$$

$$\dot{m}_{0,1,0} = - \left(\frac{k}{m}\right) m_{1,0,0} - \left(\frac{c}{m}\right) m_{0,1,0} + \left(\frac{1}{m}\right) m_{0,0,1} \qquad (2.36)$$

$$\dot{m}_{0,0,1} = - \beta m_{0,0,1}$$

On multiplying the solutions of (2.36) by $h_2(m,k,c)$ and integrating out the condition in these three conditional moments, we finally obtain the moments of x as a function of time. We obtain in analogous fashion the differential equations for the second conditional moments; we do not do this as the steps are of a mechanical nature and not of direct interest. The main point to notice is that differential equations for the conditional moments of x can be obtained when an external force is present in the equations of motion provided this force is produced by passing white noise through a suitable filter.

It is important to point out that for any gaussian external excitation the solution vector is gaussian conditioned on the random parameters. Therefore, all conditional moments can be obtained but not as easily as above [8].

The Liouville equation enabled us to obtain, in a straightforward manner, the exact expression for the conditional probability density function p. Reference to (2.34) suggests that it will be much more difficult to obtain p_1 from this equation and we shall not pursue this line of thought further.

III. Perturbation Methods

The references [9-17] address the eigenvalue (natural frequency) and eigenvector (normal mode) problem in structural systems by perturbation methods. Before discussing methods or techniques involved, it is important to understand at the outset that the geometry of the structure, how its equations of motion are assembled, the final mathematical form of the equations of motion, and how randomness in parameters is introduced have a profound influence on the nature of the results obtained.

A structure's geometry can be in the form of a linear array (chain) of elements that may, for example, consist of simple harmonic oscillators strung together in a line, beam segments continuously connected at a sequence of supports in a line, etc. The geometry is the simplest possible in such arrangements. Plate or shell type structures have a two-dimensional grid-like geometry and are next in order of complexity. Finally, we have the general case in which one or two-dimensional geometries are interconnected in a complex manner.

The equations of motion depend on the coordinate choice, particularly when the fact that mass is always distributed is taken into account. Reference [18] discusses methods of making this choice and illustrates the substantial difference in response that can occur due to different choices. Reference [16] also discusses a component mode synthesis method for selecting coordinates and assembling the equations of motion. A coordinate transformation of the equations of motion is sometimes employed as in [10,19] and the altered form of the equations may be advantageous.

Let us briefly present here a typical perturbation procedure. We consider the free motion of a conservative system governed by the equations

$$I\ddot{y} + Ky = 0 \qquad (3.1)$$

where y is the (n x 1) column vector with transpose

$$y^T = \{y_1, \ldots, y_n\} \qquad (3.2)$$

Now the elements in the symmetric stiffness matrix K are determined by the bars, beams, columns, joints, etc., making up the structure, and the uncertainties in the structure reside in these elements. Let there be m structural elements, and let the stiffness matrix of the ith structural element be

$$K_i = (1 + X_i)\underline{K_i} \quad , \quad i = 1, \ldots, m \quad , \qquad (3.3)$$

which produces the (n x n) random stiffness matrix

$$K = \sum_i^m K_i = \{K_{jk}\} \quad . \qquad (3.4)$$

The random variables $X_1, \ldots, X_m$ are regarded as small perturbation terms describing the uncertainty present in the structural elements and we assume

$$E\{X_i\} = 0 \quad , \quad \text{Var}\{X_i\} = \sigma_i^2 \quad , \qquad (3.5)$$

$\underline{K_i}$ is the mean stiffness matrix of the ith element, $K = K^T$, i.e., K is symmetric in the K_{jk}, K_{jk} is the random stiffness element corresponding to y_j and y_k, and we assume masses of the elements do not change. We note also that we can write (3.4) as

$$K = \underline{K} + \Sigma X_i \underline{K_i} \quad , \quad \underline{K} = \Sigma \underline{K_i} \qquad (3.6)$$

which gives also .

$$E\{K\} = \underline{K} \quad \text{and} \quad \frac{\partial K}{\partial X_i} = \underline{K_i} \quad , \qquad (3.7)$$

where $\underline{K}$ is the stiffness matrix of the structure with each member taking its mean stiffness.

Assume normal mode motion

$$y = \alpha \cos(\omega t + \phi) \qquad (3.8)$$

with α the (n x 1) column vector defined by

$$\alpha^T = \{\alpha_1, \ldots, \alpha_n\} \ . \tag{3.9}$$

Then, substituting (3.8) into (3.1), we obtain

$$(K - \omega^2 I)\alpha = 0 \ , \tag{3.10}$$

where again I is the $(n \times n)$ unit matrix.

The squared natural frequencies ω_r^2 are determined by the n roots of the equation

$$\det |K - \omega^2 I| = 0 \ , \tag{3.11}$$

revealing that the ω_r^2 and ω_r are random variables since K contains random variables. Let the random mode corresponding to ω_r be the $(n \times 1)$ column vector α_r. Then we can write

$$(K - \omega_r^2 I)\alpha_r = 0 \ , \tag{3.12}$$

with the usual orthogonality relations

$$\alpha_r^T I \alpha_s = 0 \ , \quad \alpha_r^T K \alpha_s = 0 \quad \text{if } s \neq r$$

$$\tag{3.13}$$

$$\alpha_r^T I \alpha_r = 1 \ , \quad \alpha_r^T K \alpha_r = \omega_r^2 \ ,$$

where "T" denotes transpose, as before.

We are now interested in expressing the random variables ω_r and α_r in terms of a power series in the random variables X_i. Consider, for example, the rth natural frequency ω_r of the system expressed in the form

$$\omega_r = \underline{\omega}_r + \sum_{i=1}^m \lambda_i X_i + \sum_{i=1}^m \sum_{j=1}^m \lambda_{ij} X_i X_j + \ldots \ , \tag{3.14}$$

where $\underline{\omega}_r$ represents the rth natural frequency of the mean system, and the $\lambda_i, \lambda_{ij}, \ldots$ are to be determined. Once we know the $\lambda_i, \lambda_{ij}, \ldots$, we can obtain statistical properties of ω_r or any other quantity of interest. Let us consider a general formulation of this problem, considering natural frequencies and normal modes. We follow the method suggested by Zarghame [14] which appears well suited to computation.

Differentiate (3.12) with respect to X_i:

$$\left[\underline{K}_i - 2\omega_r \frac{\partial \omega_r}{\partial X_i} I\right] \alpha_r + (K - \omega_r^2 I) \frac{\partial \alpha_r}{\partial X_i} = 0 \ . \tag{3.15}$$

Next premultiply (3.15) by α_r^T obtaining

$$\alpha_r^T \left[\underline{K}_i - 2\omega_r \frac{\partial \omega_r}{\partial X_i} I\right] \alpha_r = 0 \ , \tag{3.16}$$

since by the symmetry of $K(K = K^T)$ and (3.12)

$$\alpha_r^T (K - \omega_r^2 I) = 0 \ .$$

Thus, with the third of (3.13)

$$\frac{\partial \omega_r}{\partial X_i} = \frac{1}{2\omega_r} \alpha_r^T \underline{K}_i \alpha_r \ . \tag{3.17}$$

This is to be evaluated at $X_1 = ... = X_n = 0$ (i.e., $X = 0$); we obtain

$$\left(\frac{\partial \omega_r}{\partial X_i}\right)_o = \frac{1}{2\underline{\omega}_r} \underline{\alpha}_r^T \underline{K}_i \underline{\alpha}_r \ , \tag{3.18}$$

where the underbarred quantities are to be evaluated for the system with mean stiffness. We note that (3.18) gives the sensitivity coefficients [20,21] of ω_r with respect to the X_i. The importance of the sensitivity coefficients resides in the fact that they reveal by their magnitudes those ω_r that are either sensitive or insensitive to uncertainity in members values.

The α_k, $k = 1, \ldots, n$ span the coordinate space; hence, we may write

$$\frac{\partial \alpha_r}{\partial X_i} = \sum_j \beta_{ri}^{(j)} \alpha_j \ . \tag{3.19}$$

We substitute (3.19) into (3.15):

$$\left[\underline{K}_i - 2\omega_r \frac{\partial \omega_r}{\partial X_i} I\right] \alpha_r + \left(K - \omega_r^2 I\right) \sum_j \beta_{ri}^{(j)} \alpha_j = 0 \ . \tag{3.20}$$

Now premultiply (3.20) by α_k^T, obtaining

$$\alpha_k^T \left[\underline{K}_i - 2\omega_r \frac{\partial \omega_r}{\partial X_i} I\right] \alpha_r + \alpha_k^T \left(K - \omega_r^2 I\right) \sum_j \beta_{ri}^{(j)} \alpha_j = 0 \ ,$$

or, for $k \neq r$ and with the use of (3.13)

$$\left(\omega_k^2 - \omega_r^2\right)\beta_{ri}^{(k)} = -\alpha_k^T \underline{K}_i \alpha_r \quad .$$

Differentiating the next to last of (3.13) with respect to X_i gives

$$\alpha_r^T I \frac{\partial \alpha_r}{\partial X_i} = 0 \quad ,$$

which on premultiplying (3.19) by $\alpha_r^T I$ then demonstrates that $\beta_{ri}^{(r)} = 0$. Thus,

$$\beta_{ri}^{(k)} = - \frac{\alpha_k^T \underline{K}_i \alpha_r}{\omega_k^2 - \omega_r^2} \quad , \quad k \neq r \quad , \tag{3.21}$$

and hence from (3.19)

$$\frac{\partial \alpha_r}{\partial X_i} = {\sum_k}' \frac{\alpha_k^T \underline{K}_i \alpha_r}{\omega_r^2 - \omega_k^2} \alpha_k \quad , \tag{3.22}$$

where the prime on $\sum'$ means that k does not take the value r. When evaluated at $X = 0$, we have

$$\left(\frac{\partial \alpha_r}{\partial X_i}\right)_o = {\sum_j}' \frac{\underline{\alpha}_j^T \underline{K}_i \underline{\alpha}_r}{\underline{\omega}_r^2 - \underline{\omega}_j^2} \underline{\alpha}_j \tag{3.23}$$

where again the underbarred quantities are evaluated when members take on their mean stiffnesses. We note that (3.23) gives the sensitivity coefficients of the mode shapes with respect to X_j. Without showing the detailed deviation, we simply state that it can be shown that

$$\left(\frac{\partial^2 \omega_r}{\partial X_i \partial X_j}\right)_o = \frac{1}{2}\left[{\sum_{k'}} \left(\underline{\beta}_{ri}^{(k)}\beta_{kj}^{(r)} + \underline{\beta}_{rj}^{(k)}\beta_{ki}^{(r)}\right)\left(\underline{\omega}_k^2 - \underline{\omega}_r^2\right) - 2\left(\frac{\partial \omega_i}{\partial X_i}\right)_o\left(\frac{\partial \omega_i}{\partial X_j}\right)_o\right] \tag{3.24}$$

and

$$\left(\frac{\partial^2 \alpha_r}{\partial X_i \partial X_j}\right)_o = {\sum_k} \underline{\beta}_{r,ij}^{(k)} \underline{\alpha}_k \quad . \tag{3.25}$$

where

$$\underline{\beta}_{r,ij}^{(k)} = \frac{1}{\underline{\omega}_r^2 - \omega_k^2} \left[\alpha_k^T \left(\underline{K}_i - 2\underline{\omega}_i \left(\frac{\partial \omega_i}{\partial X_i} \right)_o \right) \left(\frac{\partial \alpha_r}{\partial X_j} \right)_o + \alpha_k^T \left(\underline{K}_j - 2\underline{\omega}_r \left(\frac{\partial \omega_r}{\partial X_j} \right)_o \right) \left(\frac{\partial \alpha_r}{\partial X_i} \right)_o \right], \ k \neq r \tag{3.26}$$

$$\underline{\beta}_{i,ij}^{(r)} = - \left(\frac{\partial \alpha_r^T}{\partial X_j} \right)_o I \left(\frac{\partial \alpha_r}{\partial X_i} \right)_o \tag{3.27}$$

Summarizing our results up to this point we have for the random variable

$$\omega_r = \underline{\omega}_r + \sum_i \left[\frac{\partial \omega_r}{\partial X_i} \right]_o X_i + \frac{1}{2} \sum_i \sum_j \left(\frac{\partial^2 \omega_r}{\partial X_i \partial X_j} \right)_o X_i X_j + ..., \tag{3.28}$$

where the partial derivatives are supplied by (3.18) and (3.24). We also have for the random variable

$$\alpha_r = \underline{\alpha}_r + \sum_i \left[\frac{\partial \alpha_r}{\partial X_i} \right]_o X_i + \frac{1}{2} \sum_i \sum_j \left(\frac{\partial^2 \alpha_r}{\partial X_i \partial X_j} \right)_o X_i X_j + ..., \tag{3.29}$$

where (3.23) supplies the first partial derivative, and (3.26) and (3.27) supply the derivatives in the double sum.

Let us now consider the statistics of ω_r, etc. Consider Eq. (3.28) first. We have, on taking expectation,

$$E\{\omega_r\} = \underline{\omega}_r + \frac{1}{2} \sum_i \sum_j \left[\frac{\partial^2 \omega_r}{\partial X_i \partial X_j} \right]_o E\{X_i X_j\} + ... \ . \tag{3.30}$$

Even if the X's are independent $E\{\omega_r\} \neq \underline{\omega}_r$, since the $E\{X_j^2\} \neq 0$ terms are still present. Now square (3.28) and take expectation

$$E\{\omega_r^2\} = \underline{\omega}_r^2 + \frac{1}{2} \underline{\omega}_r \sum_i \sum_j \left[\frac{\partial^2 \omega_r}{\partial X_i \partial X_j} \right]_o E\{X_i X_j\}$$

$$+ \sum_i \sum_j \left[\frac{\partial \omega_r}{\partial X_i} \right]_o \left[\frac{\partial \omega_r}{\partial X_j} \right]_o E\{X_i X_j\}$$

$$+ \frac{1}{2} \sum_i \sum_j \sum_k \left[\frac{\partial \omega_r}{\partial X_i} \right]_o \left(\frac{\partial^2 \omega_r}{\partial X_j \partial X_k} \right)_o E\{X_i X_j X_k\}$$

$$+ \sum_i \sum_j \sum_k \sum_\ell \left[\frac{\partial^2 \omega_r}{\partial X_i \partial X_j} \right]_o \left(\frac{\partial^2 \omega_r}{\partial X_k \partial X_\ell} \right)_o E\{X_i X_j X_k X_\ell\} + ... \ . \tag{3.31}$$

We can now approximate Var ω_r; it is defined as

$$\text{Var } \omega_r^2 = E\{\omega_r^2\} - [E\{\omega_r\}]^2 \; . \tag{3.32}$$

Thus, it is a straightforward task to approximate the first two moments of ω_r.

If we extend (3.28) to cubic, quartric, ... terms in the X_i, then (3.30) and (3.31) would contain additional terms. How far we should continue this process will depend on the relative size of the terms containing $E\{X_iX_j\}$, $E\{X_iX_jX_k\}$, etc. and what information we have that would enable us to evaluate these expectations. It is not usual that we can evaluate any more than $E\{X_iX_j\}$.

We note that Zarghame's method described above gives sensitivity coefficients for natural frequencies and corresponding normal modes plus series expansions for these quantites in terms of the random variables $X_1, \ldots, X_m$ that define the uncertainty present in the stiffness matrix K. Moments of the quantities are easily obtained, but it is practically impossible to obtain distributions for the natural frequencies and corresponding normal modes. For confidence interval location and size for a natural frequency, for example, we must approximate using

$$E\{\omega_r\} \pm 3 \sqrt{\text{Var } \omega_r} \tag{3.33}$$

as a rough indication of a 99% confidence interval. This interval gives us some idea of the spread in a natural frequency and it could be employed to make reasonably sure that no steady excitation frequencies were contained therein for all ω_r. Alternatively, we might employ the signal to noise ratio

$$\frac{E\{\omega_r\}}{\sqrt{\text{Var } \omega_r}} \tag{3.34}$$

to obtain an idea of how important stiffness uncertainty is for natural frequency; if (3.34) is greater than 20 or 30 say, we would regard the location of ω_r as deterministic; on the other hand, if (3.34) is less than 5-10, it might be unwise to ignore this level of variability in the location of ω_r, depending, of course, on the consquences of such uncertainty.

IV. Mean Square Approximate Systems

We consider, in this section, a technique for including disorder or parameter uncertainty that follows a different line than taken in previous sections. Specifically, mean square systems are employed. We begin by introducing this type of system [22,23].

Let us begin with a very simple example in which there is no disorder and no damping. Let the coordinates $q_1, \ldots, q_n$. Then, with

$$2T = m_{jk}\dot{q}_j\dot{q}_k, \quad 2V = k_{jk}q_jq_k, \quad \delta W = f_j(t)\delta_{qj} \quad , \tag{4.1}$$

where summation is on multiple subscripts. Then, with mass coefficients included, (4.1) can be rewritten as

$$m_{jk}\ddot{q}_k + k_{jk}q_k = f_j(t) \quad . \tag{4.2}$$

Let, with f_{oj} constant,

$$f_j(t) = f_{oj}\cos(\omega t + \phi) \quad . \tag{4.3}$$

Then, the forced motion

$$q_k = u_k\cos(\omega t + \phi) \tag{4.4}$$

satisfies

$$(k_{jk} - \omega^2 m_{jk}) u_k = f_{oj} \quad . \tag{4.5}$$

These equations state that given the f_{oj} and ω, the u_k are determined by the solution of this linear system of equations. Further, if ω is the natural frequency ω_r and the u_k define the rth mode shape α_{rk}, then the f_{oj} must vanish. Let us look at the natural frequency problem in an unorthodox manner.

Suppose we pick an ω and a set of u_k which may not be one of the natural frequencies and normal modes. Then the right of (4.5) will not be zero and we need force amplitudes ϵ_j to produce this motion:

$$(k_{jk} - \omega^2 m_{jk}) u_k = \epsilon_j \quad . \tag{4.6}$$

The ϵ_j are the amplitudes required to maintain the assumed motion; we regard the ϵ_j as the amplitudes of the constraint forces required to produce the motion.

Consider next a summation of second order amplitudes

$$I(n,\omega) = \sum_1^n \epsilon_j^2 > 0 \quad . \tag{4.7}$$

For a fixed ω, this is a positive definitive quadratic function of the u's. We can use this equation to find the natural frequencies ω_r and corresponding normal modes α_{rk}. Assume the u's are normalized in some manner (for example, $u_n = 1$, or better $m_{jk} u_j u_k = 1$). For fixed ω, we find the minimum of $I(u,\omega) > 0$. Notice that if $\omega = \omega_r$ the u's that produce a minimum are the α_{rk} and $I(\alpha_{rk}, \omega_r^2) = 0$, since the $\epsilon_j = 0$, $j = 1, \ldots, n$, in this case. It follows that if for a specified ω we find the minimum of $I(u, \omega^2)$ and this minimum equals zero, then this ω is a natural frequency and the u that produces this zero minimum are proportional to the corresponding normal mode. Let us consider another interesting aspect of this method.

Consider a frequency window $g(\omega)$ with the following properties:

$$g(\omega) > 0 \quad , \quad \omega' < \omega < \omega'' \quad ,$$

$$\int_{\omega'}^{\omega''} g(\omega) d\omega = 1 \quad , \quad \int_{\omega'}^{\omega''} \omega^2 g(\omega) d\omega < \infty \quad . \tag{4.8}$$

Replace (4.7) with

$$I(u,g) = \int_{\omega'}^{\omega''} \sum \epsilon_j^2 d\omega \quad . \tag{4.9}$$

Find the the u that makes (4.9) a minimum. The interesting feature of this method is that if there is a natural frequency of the system in the frequency interval (ω', ω''), the u in $\min I(u,g)$ determine the normal mode of this natural

frequency. Let these u be in component form $\{u_1^{(r)}, \ldots, u_n^{(r)}\}$; then the corresponding natural frequency is determined by the Rayleigh quotient:

$$\omega_r^2 = \frac{k_{jk} u_j^{(r)} u_k^{(r)}}{m_{jk} u_j^{(r)} u_k^{(r)}} \quad , \tag{4.10}$$

where we assume we have found the rth normal mode and its natural frequency. It follows that if there is concern that an interval (ω', ω'') contains a natural frequency, we have the method for determining if this is the case without determining all natural frequencies of the system. References [22,23] give details on this matter we will not discuss in this report.

The computational problem of finding the minimum of $I(u, \omega^2)$ is carried out using one of a number of computer codes based upon conjugate gradient techniques, and, hence, is not a problem.

So far, there has been no disorder in our system; i.e., the parameters m_{jk} and k_{jk} have been assumed to take definite values. Let us assume at this point that mass and stiffness contain random variables. We define, in this case,

$$I(u, g) = E \left\{ \int_{\omega'}^{\omega''} g(\underline{\omega}) \sum_1^n \epsilon_j^2 d\underline{\omega} \right\} \quad , \tag{4.11}$$

where, in vector-matrix form

$$\sum_1^n \epsilon_j^2 = u^T (K - \underline{\omega}^2 M)^T (K - \underline{\omega}^2 M) u \quad . \tag{4.12}$$

Since E only operates on $\sum_1^n \epsilon_j^2$ in (4.11), we have

$$E \left\{ \sum_1^n \epsilon_j^2 \right\} = E \left\{ u^T (K - \underline{\omega}^2 M)^T (K - \underline{\omega}^2 M) u \right\} \quad , \tag{4.13}$$

and ω is a fixed parameter in (4.13). We assume the u are parameters to be determined. Thus, (4.13) takes the form

$$E\left\{\sum_1^n \epsilon_j^2\right\} = u^T E\left\{(K-\underline{\omega}^2 M)^T(K-\underline{\omega}^2 M)\right\} u \ . \qquad (4.14)$$

We note that $(K-\underline{\omega}^2 M)^T = K-\underline{\omega}^2 M$ because of the symmetry assumed for K and M. In all events, means and second moments of K and M are all the information needed to determine the expectation in (4.14).

We than proceed as in the deterministic case, since I(u,g) has a deterministic form.

To relate (4.7) to (4.13), all we have to do is assume

$$g(\underline{\omega}) = \delta(\omega-\underline{\omega}) \ , \qquad (4.15)$$

where $\delta(.)$ is the delta function. The substitution of (4.15) into (4.11) yields

$$I(u,\omega) = u^T E\left\{(K-\omega^2 M)^T(K-\omega^2 M)\right\} u \ . \qquad (4.16)$$

This expression differs from (4.7) because of the assumed random parameters in K and M. If in (4.7), ω is a natural frequency of the deterministic system, $I(u,\omega)=0$. The $I(u,\omega)>0$ in (4.16) because of the random parameters. Use of this fact has been made in [32] to obtain an estimate of the variance of natural frequency ω_r; the formula is

$$\text{Var } \omega_r = \frac{I(u_r,\omega_r^2)}{4\omega_r^2} \ , \qquad (4.17)$$

where ω_r is the rth natural frequency and u_r is the corresponding normal mode for the system with mean parameter values. Monte Carlo simulation [24] reveals that (4.17) can be conservative and a correction is suggested. Equation (4.17) is easy to use since a minimum for I is not required. Further, (4.17) provides a much simpler method for estimating the variance of ω_r than given in Section III. However, mean square approximate systems do not provide any information on $E\{\omega_r\}$ or on variability in mode shape. Let us next consider how these systems apply to estimating frequency response with parameter

uncertainty present.

We take the equations of motion in the form

$$M\ddot{q} + C\dot{q} + Kq = f \quad . \tag{4.18}$$

The frequency response $Z^{-1}(\omega)$ and $Z(\omega)$ are defined as follows. For the external force,

$$f = \delta_{jr}e^{i\omega t} \quad , \tag{4.19}$$

with r fixed at $\delta_{jr}=0$ for $j{\neq}r$, $\delta_{rr}=1$, the component form of (4.18) is

$$q_j = Z_{jr}^{-1}(\omega)e^{i\omega t} \tag{4.20}$$

which is exact.

Suppose we try to approximate (4.20) with

$$q_j = \beta_{jr}e^{i\omega t} \quad , \tag{4.21}$$

where the β_{jr} are not known in advance. The equations of motion now are not satisfied and we must introduce constraint forces ϵ_j to bring about their satisfaction as

$$\left(K_{jk}- \omega^2 M_{jk} + i\omega C_{jk}\right) \beta_{kr}- \delta_{jr} = \epsilon_{jr} \quad . \tag{4.22}$$

From

$$I(\beta,\omega) = E \left\{ \sum_1^n \epsilon_{jr}^* \epsilon_{jr} \right\} \quad , \tag{4.23}$$

where asterisk denotes complex conjugate. This I is just like (4.11) except the β have replaced the u. We find the β that make (4.23) a minimum, denote this β by $\hat{\beta}$. Then, $\hat{\beta}=\{\hat{\beta}_1, \ldots, \hat{\beta}_n\}$ is the mean square approximate to the $Z_{jr}^{-1}(\omega)$. It can be shown that if the system is deterministic (i.e., contains no random parameters) the $\hat{\beta}$ are exactly the $Z_{jr}^{-1}(\omega)$.

The ϵ_{jr} are complex, hence, the right of (4.23), when written out, is

$$\sum_{j=1}^n E \left\{ \left[\left(K_{jk}- \omega^2 M_{jk}- i\omega C_{jk}\right)\beta_{kl}^* \delta_{jr} \right] \left[\left(K_{jl}- \omega^2 M_{jl}+i\omega C_{jl}\right)\beta_{lr}- \delta_{jr} \right] \right\} \quad . \tag{4.24}$$

It follows that the minimum of (4.23) is for the real and imaginary parts of β_{kr}. This added complication poses no additional computational problem [25,26].

The method also supplies an error criterion that makes it possible to judge the accuracy of the β_{kr}.

References [25,26] describe in some detail how the above technique can be applied to estimating the frequency response in a number of structures with specific attention being paid to numerical details of the computations. Reference [1] also describes how these techniques can be applied to the construction of a sequence of approximants for a complex system by starting from a highly constrained initial system and gradually relaxing the constraints. In these three references, extensive use is made of the error criterion to determine when the estimated quantities (usually frequency response) are sufficiently accurate for the purpose in hand. A comment on what mean square approximate system provide is now in order.

We observe, for example, that these systems enable us to estimate frequency response $Z_{kr}^{-1}(\omega)$ by means of $\beta_{kr}(\omega)$. The $\beta_{kr}(\omega)$ are deterministic numbers that take into account the means and variances of the statistical parameters of the structure. Thus, the $\beta_{kr}(\omega)$ provide a deterministic estimate for $Z_{kr}^{-1}(\omega)$. In the form given above and in [25,26,27], it is not possible to obtain statistical information concerning the $Z^{-1}(\omega)$. However, it is possible to employ Monte Carlo methods to obtain estimates for the $\beta_{kr}(\omega)$ given the parameters are sample values to obtain sample values for the $\beta_{kr}(\omega)$ from which statistical information can be obtained.

The statistical energy approach (SEA) merits mention at this point since it also employs average energy concepts [28-32]. Basically, SEA estimates the average flow of energy from one part of a structure to another. For example, if

there is energy input into one part of a complex structure, this method provides an estimate of how this energy flows into another part of the structure. Thus, it is possible to estimate average vibrational energy present in any part of the structure. Information of this type is frequently the only type of information it is possible to obtain about the response in an extremely complex structure containing a large number of undamped natural frequencies in 1 Hertz. Insofar as we know, nothing has yet been done to include the influence of statistical parameters; however, the work given in [31] suggests it might be possible to do this.

V. Statistical Linearization of Nonlinear Systems

1. Systems with Known Nonlinearity

In a number of real systems the non-linearity property may be known. The statistical properties of the response of such systems cannot, in general, be determined. However, the statistical properties of the response of its linearized version may result in an adequate approximation. To obtain the linearized form, the method of statistical linearization has often been applied [33].

The basic idea is generated as follows. Consider the true system model

$$\dot{y}(t) + f(y(t)) = n(t) \tag{5.1}$$

where $n(t)$ is the random excitation vector, which is assumed to be known as well. The idea is to approximate (5.1) as the linear form

$$x(t) + A x(t) = n(t) \tag{5.2}$$

The approach is to write (5.1) as,

$$\dot{y}(t) + A y(t) + \left[f(y(t)) - A y(t) \right] = n(t) \tag{5.3}$$

The linearization is obtained by choosing matrix A so that the ensemble average of the mean of the difference terms in (5.3) is minimized, i.e.,

$$\min_{A} E \left\{ \left\| f(y(t)) - A y(t) \right\|^2 \right\} \tag{5.4}$$

The solution is simply obtained as

$$A = E \left\{ f(y) y' \right\} E \left\{ y y' \right\}^{-1} \tag{5.5}$$

where $(\ )'$ denotes transpose.

Therefore, A is defined in terms of the y-statistics. But, the y-statistics are unknown! If they were known, we would not have to linearize the model. The traditional idea is to determine A via the linear X-system statistics as

$$A = E \left\{ f(x) x' \right\} E \left\{ x x' \right\}^{-1} \tag{5.6}$$

The statistics in (5.6) can be exactly obtained in general, but when n is

Gaussian its determination defined by (5.2) becomes simpler. It should be pointed out that if f(y) is a vector polynomial then (5.5) can be determined exactly as statistical values, through stationary solutions of various forms [34].

The following example for the Duffing oscillator with white noise excitation is very typical.

Example I- Statistical linearization for the Duffing oscillator with white noise excitation.

Consider the second order system

$$\ddot{y}(t) + \dot{y}(t) + y^3(t) = \dot{B}(t) \tag{5.7}$$

The $\dot{B}$ process is the gaussian white noise, given through the classic Brownian process, which is Gaussian. The classic Ito form of (5.7) is written as

$$\begin{cases} dy_1 = y_2 dt \\ dy_2 = -(y_2+y_1^3)dt + dB \end{cases} \tag{5.8}$$

From the conditions of B, we have

$$E\{dB\} = 0 \quad , \quad E\{(dB)^2\} = \sigma^2 dt \tag{5.9}$$

We wish to determine the linear form

$$\ddot{x}(t) + \dot{x}(t) + kx(t) = \dot{B}(t) \tag{5.10}$$

which we can write as

$$\begin{cases} dx_1 = x_2 dt \\ dx_2 = -(x_2+kx_1)dt + dB \end{cases} \tag{5.11}$$

Clearly, k is determined as

$$k = E\{x_1^3 \cdot x_1\} E\{x_1^2\}^{-1} = E\{x_1^4\} E\{x_1^2\}^{-1} \tag{5.12}$$

However, since (x_1, x_2) is gaussian, then we find

$$k = 3E\{x_1^2\} \tag{5.13}$$

For the system (5.11) we can determine the stationary density as

$$p(x_1, x_2) = \frac{\sqrt{k}}{\pi \sigma^2} \exp\left[-\frac{1}{\sigma^2}(x_2^2 + kx_1^2)\right] \qquad (5.14)$$

This allows us to obtain the value of k and the x-moments as

$$k = 1.22\,\sigma; \ E\{x_1 x_2\} = 0 \ ; \ E\{x_1^2\} = 0.408\,\sigma \ ; \ E\{x_2^2\} = 0.500\,\sigma^2 \qquad (5.15)$$

It is interesting to point out that for this particular example the moments of (y_1, y_2) can also be obtained for the stationary case as

$$E\{y_1 y_2\} = 0 \ ; \ E\{y_1^2\} = 0.478\,\sigma \ ; \ E\{y_2^2\} = 0.500\,\sigma^2 \qquad (5.16)$$

We see that the error in the linear form is only in $E\{x_1^2\}$, and is

$$\frac{E\{y_1^2\} - E\{x_1^2\}}{E\{y_1^2\}} = \frac{0.070}{0.478} \ \text{or} \ 14.4\% \qquad (5.17)$$

This is not an unacceptable error.

2. Systems with Stochastic Parameters

The question that must be considered here is, if the system has not only nonlinearities, but also possesses parameters that are random as well, in what way may we apply statistical linearization so that the random constant coefficients of the true system are reflected in the approximate system. We illustrate the procedure by considering again the Duffing equation

$$\ddot{y} + c\dot{y} + by^3 = \dot{B}(t) \qquad (5.18)$$

where here b is assumed to be a random constant. The linear form is taken as

$$\ddot{x} + c\dot{x} + \alpha bx = \dot{B}(t) \qquad (5.19)$$

where α, the linearization constant, is determined from

$$\min_{\alpha} E\left\{ \|by^3 - \alpha by\|^2 \right\} \qquad (5.20)$$

which leads to

$$\alpha = E\{b^2 y_1^4\}\, E\{b^2 y_1^2\}^{-1} \qquad (5.21)$$

In the terms of conditional moments (5.18) can be expressed as

$$\alpha = E\left\{b^2\ E\{y_1^4\,|b\}\right\} E\left\{b^2 E\{y_1^2\,|b\}\right\}^{-1} \qquad (5.22)$$

Following the traditional path, we determine α from the linear x-system statistics, so that

$$\alpha = E\left\{b^2 E\{x_1^4\,|b\}\right\} E\left\{b^2 E\{x_1^2\,|b\}\right\}^{-1}$$

$$= E\left\{b^2\left[E\{x_1^2\,|b\}\right]^2\right\} E\left\{b^2\{x_1^2\,|b\}\right\}^{-1} \qquad (5.23)$$

The conditional second moments are readily obtained as

$$E\{x_1 x_2\,|b\} = 0\ ; \quad E\{x_1^2\,|b\} = \frac{\sigma^2}{2\alpha bc}\ ; \quad E\{x_2^2\,|b\} = \frac{\sigma^2}{2c} \qquad (5.24)$$

which, when substituted into (5.23), leads to

$$\alpha^2 = \frac{3}{2}\frac{\sigma^2}{c}\frac{1}{E\{b\}} = \frac{3}{2}\frac{\sigma^2}{c\overline{b}} \qquad (5.25)$$

when the mean of b, $E\{b\}=\overline{b}$. The x-moments can then be determined

$$E\{x_1^2\} = E\left\{E\{x_1^2\,|b\}\right\} = 0.408(\overline{b}/c)^{1/2}\sigma\ E\{1/b\}$$

$$E\{x_2^2\} = E\left\{E\{x_2^2\,|b\}\right\} = 0.500\,\sigma^2/c \qquad (5.26)$$

For illustration, assume a uniform distribution function for the opening constant b with mean $\overline{b}$,

$$p(b) = \begin{cases} 1/2\epsilon\overline{b}\ , & \overline{b}(1-\epsilon)\leq b\leq\overline{b}(1+\epsilon) \\ 0 & ,\ \text{otherwise} \end{cases} \qquad (5.27)$$

Thus

$$E\{1/b\} = \frac{1}{2\epsilon\overline{b}}\ \ell n\ \frac{(1+\epsilon)}{(1-\epsilon)} \qquad (5.28)$$

where ϵ=bandwidth parameters. Hence,

$$E\{x_1^2\} = 0.204(1/\overline{b}c)^{1/2}\ (\sigma/\epsilon)\ell n\ \frac{(1+\epsilon)}{(1-\epsilon)} \qquad (5.29)$$

The exact solution for this case is

$$E\{y_1^2\} = 0.478(1/\overline{bc})^{1/2}(\sigma/\epsilon) \ [(1+\epsilon)^{1/2} - (1-\epsilon)^{1/2}] \tag{5.30}$$

and the error is

$$\frac{E\{y_1^2\} - E\{x_1^2\}}{E\{y_1^2\}} = \frac{0.478\ [(1+\epsilon)^{1/2} - (1-\epsilon)^{1/2}] - 0.204\ \ell n(1+\epsilon/1-\epsilon)}{0.478[(1+\epsilon)^{1/2} - (1-\epsilon)^{1/2}]} \tag{5.31}$$

which for $\epsilon = 0$ is 14.5%.

VI. Closure

The requirement for precision pointing of large scale structures, consisting of trusses, radio reflectors and optical systems, has motivated the investigation of how parametric uncertainties affect the system response characteristics.

Randomness of these large structures may arise from many sources, for example: manufacturing processes; space assembly by humans, variations in environmental conditions that may bear directly upon the material properties and resulting mechanical behavior. These uncertainities must be accounted for on some rational basis so that the quantities of concern for describing the response of the structure can be statistically characterized. Such quantities include: natural frequencies, normal modes, frequency response, etc. Specific random techniques that are available and discussed in this report are: Liouville's equation, perturbation methods, mean-square approximate systems, and statistical linearization.

These are the techniques that we shall consider in order to develop procedures for determining the response of space structures. It is conceivable, however, that advances in these techniques will have to be developed as our research effort unfolds.

VII. References

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