NASA Contractor Report 177556

Extra Compressibility Terms for Favre-Averaged Two-Equation Models of Inhomogeneous Turbulent Flows

Morris W. Rubesin

MCAT Institute
Ames Research Center
Moffett Field, California

Prepared for
Ames Research Center
CONTRACT NCC2-585
June 1990

National Aeronautics and
Space Administration

Ames Research Center
Moffett Field, California 94035-1000

EXTRA COMPRESSIBILITY TERMS FOR FAVRE-AVERAGED TWO-EQUATION MODELS OF INHOMOGENEOUS TURBULENT FLOWS

Morris W. Rubesin

MCAT Institute
NASA Ames Research Center
Moffett Field, CA 94035

ABSTRACT

Forms of extra compressibility terms that result from use of Favre averaging of the turbulence transport equations for kinetic energy and dissipation are derived. These forms introduce three new modeling constants, a polytropic coefficient that defines the interelationships of the pressure, density, and enthalpy fluctuations and two constants in the dissipation equation that account for the non-zero pressure-dilitation and mean pressure gradients.

NOMENCLATURE

$\tilde{a} = mean\ velocity\ of\ sound$

$c_e = modeling\ coefficient,\ Reynolds\ heat\ flux,\ Eq.\ (13)\ [0.35]$

$C_\mu = modeling\ coefficient,\ eddy\ viscosity\ [0.09]$

$h = enthalpy$

$H = total\ enthalpy,\ Eq.\ (5)\ or\ Eq.\ (9)$

$k = turbulence\ kinetic\ energy$

$\quad k'' = (u_i''u_i'')/2$

$\quad \tilde{k} = \overline{\rho u_i''u_i''}/(2\bar{\rho})$

$M_e = free-stream\ Mach\ number$

$n = polytropic\ coefficient,\ Eq.\ (10)$

$p = pressure$

$q_j = molecular\ heat\ flux\ in\ the\ jth\ direction, -(\tilde{\mu}/Pr)h_j$

$Q_i = Reynolds\ heat\ flux\ in\ ith\ direction$

$S_{ij} = strain\ rate\ tensor,\ 1/2(u_{i,j} + u_{j,i} - (2/3)\delta_{ij}u_{k,k})$

$T_w = surface\ temperature$

$T_{0e} = free-stream\ stagnation\ temperature$

T_{ij} = *total stress tensor, viscous plus Reynolds stress*
u_j = *velocity in jth direction*

β = *fluctuating density variance,* $\sqrt{\overline{\rho'^2}}/\overline{\rho}$
δ_{ij} = *Kronecker delta*
$\tilde{\epsilon}$ = *dissipation rate of kinetic energy*
μ = *molecular viscosity*
ρ = *density*
τ_{ij} = *viscous stress,* $2\tilde{\mu}S_{ij}$

$\overline{A}$ = *ensemble mean variable*
A' = *fluctuating variable*
$\tilde{A}$ = *mass weighted ensemble mean variable*
A'' = *fluctuating variable in mass weighted expansion*
$A_{,t}$ = *partial derivative with respect to time*
$A_{,i}$ = *partial derivative with respect to the ith coordinate*
$< A > = \sqrt{\overline{A^2}}$

INTRODUCTION

Favre, or mass-weighted, averaging, Ref. 1, has become the most popular approach for establishing the field equations used in evaluating compressible turbulent flows. One reason for this is that the equations for mass, momentum and energy conservation retain the relative simplicity, term for term, of their incompressible counterparts, Ref 2. The effects of density fluctuations are submerged within the definitions of the Reynolds stresses and heat or mass fluxes and the profusion of new terms that occurs when primitive variables are employed, e.g. Ref. 3, is avoided. The effects of density fluctuations, however, become reintroduced into the problem when use is made of field equations representing the turbulence moments and scale. These equations contain additional terms unique to compressible flows that involve density fluctuations and the non-zero divergence of the turbulent fluctuating velocities. The purpose of this paper is to reexamine the modeling these compressibility terms, with emphasis being given to the terms that appear in two-equation models of turbulence. Of course, similar forms of these compressibility terms can be applied to second-order closure models as well.

The possible importance of these compressibility terms was first suggested in the paper by Wilcox and Alber, Ref. 4. These authors modeled the aggregate of these terms in the turbulence kinetic energy equation with an analysis that employed the assumptions that pressure and total temperature fluctuations can be neglected. They developed a model for these terms, expressed in mean variables, that introduced a single additional modeling coefficient. No corrections were applied to the scale equation. The turbulence model, with a particular modeling coefficient, was applied to some selected supersonic flows, including near wake boundary layers with Mach numbers up to 8. The computed results agreed resonabley well with the experimental data, but the absence of computed results without the additional compressibility term in this reference did not permit firm conclusions to be drawn by a reader as to the value of including this term.

Oh, Ref. 5, reexamined these compressibility terms in the turbulence kinetic energy equation with the view of improving computations of the spread rate of a free-shear layer formed by a single supersonic stream. As in Ref. 4, no modifications were made to the scale of turbulence, which, in this case, was prescribed algebraically. Oh's analysis of the compressibility effects was based on an assumed detailed structure of two-dimensional shocklets developed by the turbulent eddies in relative supersonic motion. In addition, a form of Taylor's hypothesis for the turbulent motions was invoked. Although, the process introduced two new modeling coefficients, one could be neglected by Oh as it was a factor of the term involving the mean pressure gradients, which were set to zero in the free-shear problem. The second modeling coefficient was shown to produce a diminished spread rate with increased Mach number, as occurs within the experimental data. It was found, however, that good agreement with the data over the range of Mach number from 1 to 4 would have required altering the value of the coefficient by about 40% over the Mach number range.

To avoid conjectures, unsupported by experimental data, regarding the detailed structure of shock waves created by colliding three-dimensional eddies, Rubesin, Ref. 6, derived formulations for the extra compressibility terms in the turbulence kinetic energy equation by introducing the concept that the state conditions within an eddy could be related polytropically. This was another way of accounting for non-isentropic processes, shocklets, etc., that occur within the turbulence, but without introducing structural assumptions. It was necessary, however, to make assumptions regarding the fluctuation of the total temperature and the variance of the density fluctuations. In Ref. 6, it was assumed that the total temperature did not fluctuate and that the variance of the density fluctuations varied very slowly along streamlines. The resulting formulations for the compressibility terms in the kinetic energy equation contained a single modeling coefficient, namely, the polytropic coefficient.

The work of Ref. 6, was extended by Vandromme, Ref. 7, to retain the effects of large mean density variations across streamlines and to introduce extra compressibility terms into the turbulence scale equation, i.e. the rate of dissipation of turbulence kinetic energy, as well as in the kinetic energy equation. In addition to the polytropic coefficient, Vandromme introduced a modeling coefficient in the added term in the dissipation rate equation. Vandromme found that the spread rate of the free-shear layer in supersonic flow could be represented very well over the Mach number range from 0 to 4 with single values of the polytropic coefficient and the modeling coefficient in the dissipation equation.

There have been several papers addressed to the evaluation of compressibility effects within internal combustion engines, e.g. Refs. 8-10. In Refs. 8 and 9, the burden of maintaining reasonable length scale behavior with a two-equation model during volumetric compressions was placed on the dissipation equation, with the result that an an additional divergence term was added to the dissipation equation in k-ϵ turbulence modeling. The turbulence kinetic energy equation was left without additional terms, which, as argued in

Ref. 10, is consistent with the low Mach numbers characteristic of internal combustion engines. The difference, then, within this group of papers is in the values chosen for the modeling coefficient multiplying the divergence term.

In a recent paper, Ref. 11, Zeman identified the " dilitation dissipation" as the extra compressibility term in the kinetic energy equation. In a manner similar to Oh, Zeman utilized a modeled shock wave structure within the turbulent eddies to obtain the functional form of the compressibility correction. Assumptions regarding the shock wave thickness relative to the eddies and the probability distribution function of the turbulence Mach number led to a form of the compressibility term that contained two new modeling coefficients, one of them being the kurtosis of the turbulence. It was shown that appropriate constant values of these modeling coefficients in computations of the behavior of a free-shear layer could be made to represent the data of Ref. 12 quite well.

From this brief review, it is apparent that various authors chose considerably different routes to introduce compressibility terms into the field equations for either the turbulent kinetic energy equation or the dissipation equation, or both. Under high Mach number conditions, the order of magnitude arguments in Ref. 10 would suggest that these terms should be included in both equations.

Because of the flexibility introduced into these models by the introduction of one or more additional modeling coefficients, agreement with particular experimental data, for example, high-speed free-shear experiments, could be achieved by several disparate models, Refs. 5, 7, and 11. In the absence of the application of these same models to more complex flow situations, such as flows where shock waves impinge on boundary layers to produce separation or where they interact with shear layers away from surfaces, it is not known if the various models will differentiate and show one that is superior. To perform such a comparative study, it will be best to utilize a single computer code that contains a series of the alternative turbulence models cited here which can be turned on and off at will.

The present paper is devoted to the development of one such model. It is an extension of the work of Refs. (6) and (7), but with particular attention being given to the non-adiabatic character of high-Mach number flow fields and their large variations in the fluid properties. In addition, the model allows inhomogenieties in the turbulence.

ANALYSIS

When Favre or mass-weighted ensemble averaging is applied to the conservation equations describing the turbulent flow of a compressible fluid, the following set of equations results. The only physical assumptions applied to these equations are that the fluctuations in the fluid viscosity and thermal conductivity can be ignored. Turbulence effects on molecular properties are not expected to be important where the molecular properties themselves dominate.

Continuity

$$\bar{\rho}_{,t} + (\bar{\rho}\tilde{u}_j)_{,j} = 0 \tag{1}$$

Momentum in the ith direction

$$(\bar{\rho}\tilde{u}_i)_{,t} + (\bar{\rho}\tilde{u}_j\tilde{u}_i + \delta_{ij}\bar{p} - T_{ij})_{,j} = 0 \tag{2}$$

Here, the total stress is composed of molecular and turbulence contributions.

$$T_{ij} = \tilde{\tau}_{ij} + \overline{\tau''_{ij}} - \overline{\rho u''_i u''_j} \tag{3}$$

Total energy

$$(\bar{\rho}\tilde{H} - \bar{p})_{,t} + (\bar{\rho}\tilde{u}_j\tilde{H})_{,j} =$$

$$-\left[\overline{\rho u''_j h''} + \overline{\rho u''_j k''} + \tilde{q}_j + \overline{q''_j} - \tilde{u}_i T_{ij} - \overline{u''_i}\tilde{\tau}_{ij} - \overline{u''_i \tau''_{ij}}\right]_{,j} \tag{4}$$

The quantity $\tilde{H}$ represents the mean total enthalpy, which is the sum of the mean static enthalpy, $\tilde{h}$, and the kinetic energies of the mean motion plus the turbulence.

$$\tilde{H} = \tilde{h} + \tilde{u}_i\tilde{u}_i/2 + \overline{\rho u''_i u''_i}/(2\bar{\rho}) \tag{5}$$

In addition, the turbulence kinetic energy equation can be written as

$$\begin{aligned}(\bar{\rho}\tilde{k})_{,t} + (\bar{\rho}\tilde{u}_j\tilde{k})_{,j} = {} & -(\overline{\rho u''_i u''_j})\tilde{u}_{i,j} - (\overline{\rho u''_j k''} + \overline{u''_j p'} - \overline{u''_i \tau''_{ij}})_{,j} \\ & - \overline{u''_{i,j}\tau''_{ij}} + \overline{u''_i}\tilde{\tau}_{ij,j} + \overline{p' u''_{i,i}} - \overline{u''_i}\bar{p}_{,i}\end{aligned} \tag{6}$$

The additional compressibility terms in these equations, absent for incompressible fluids, are those involving $\overline{u''_i}$, $\overline{\tau''_{ij}}$, $\overline{q''_j}$ and $\overline{u''_{i,i}}$.

Before evaluating the quantities that contain these additional compressibility terms, it is worthwhile to eliminate those of little consequence. In regions where the molecular transport properties are important, it is to be expected that the effects of density fluctuations should be insignificant. With this assumption, the terms $\overline{\tau''_{ij}}$, $\overline{q''_j}$, $\overline{u''_i}\tilde{\tau}_{ij}$, and $\overline{u''_i}\tilde{\tau}_{ij,j}$ can be omitted from Eqs. (3), (4), and (6).

These equations then reduce to

$$T_{ij} = \tilde{\tau}_{ij} - \overline{\rho u''_i u''_j} \tag{7}$$

$$(\bar{\rho}\tilde{H} - \bar{p})_{,t} + (\bar{\rho}\tilde{u}_j\tilde{H})_{,j} =$$

$$-\left[\overline{\rho u''_j h''} + \overline{\rho u''_j k''} + \tilde{q}_j - \tilde{u}_i T_{ij} - \overline{u''_i \tau''_{ij}}\right]_{,j} \tag{8}$$

and

$$(\bar{\rho}\tilde{k})_{,t} + (\bar{\rho}\tilde{u}_j\tilde{k})_{,j} = -(\overline{\rho u_i'' u_j''})\tilde{u}_{i,j} - (\overline{\rho u_j'' k''} + \overline{u_j'' p'} - \overline{u_i'' \tau_{ij}''})_{,j}$$
$$-\overline{u_{i,j}'' \tau_{ij}''} + \overline{p' u_{i,i}''} - \overline{u_i''}\bar{p}_{,i} \tag{9}$$

The first step required in the modeling of the remaining additional terms is to determine a relationship between the fluctuations of velocities and the fluctuations of the state properties of the fluid. This is initiated by following Ref. 6 and assuming that the fluid behaves in a polytropic manner. Thus, the polytropic coefficient, n, becomes a turbulence modeling parameter.

Expressed in mass weighted variables, this assumption becomes

$$\frac{p'}{\bar{p}} = n\frac{\rho'}{\bar{\rho}} = \frac{n}{n-1}\frac{\rho T''}{\bar{\rho}\tilde{T}} = \frac{n}{n-1}\frac{\rho h''}{\bar{\rho}\tilde{h}} \tag{10}$$

The last relationship is only exact for constant values of the specific heat, but will be used here as a first approximation even when the specific heat does vary.

Next, a relationship between T'' or h'' and u_i'' has to be established. In Ref. 6, it was assumed that the total enthalpy at a point in space remained constant with time despite the turbulence. This is expressed as

$$H'' = h'' + \tilde{u}_i u_i'' + (\frac{u_i'' u_i''}{2} - \tilde{k}) = 0 \tag{11}$$

If the higher order terms are neglected, then

$$h'' = -\tilde{u}_i u_i'' \tag{12}$$

The experimental results cited in Ref. 13, however, indicate that within an hypersonic boundary layer, e.g. at $M_e = 6.4$ over a cooled surface where $T_w/T_{0e} = 0.46$, the local r.m.s. total temperature fluctuations are not zero, but can have magnitudes ranging from

$$1.4\% < \frac{< H'' >}{\tilde{H}} < 4.8\%$$

with the larger variances occurring closer to the surface. Equation (12), therefore, has to be modified to account for fluctuations in the total temperature.

One could expect that the local temperature or enthalpy fluctuation would be dependent on the intensity of the turbulence and, also, on the mean static temperature or enthalpy gradients. One such form, analagous to a mixing length formulation, is

$$h'' = -\alpha u_i'' \tilde{h}_{,i} \tag{13}$$

where α is a local mean parameter, yet to be determined.

If Eq. (13) is multiplied by $\rho u_j''$ and averaged, there results

$$Q_j = \overline{\rho u_j'' h''} = -\alpha \overline{\rho u_j'' u_i''}\tilde{h}_{,i} \tag{14}$$

which relates the local heat flux to the Reynolds stresses and the mean enthalpy gradients. This form is consistent with the modeled forms developed in Ref. 14, namely

$$Q_j = -c_e \frac{\tilde{k}}{\tilde{\epsilon}} \overline{\rho u_j'' u_i''} \tilde{h}_{,i} \tag{15}$$

with

$$c_e = 0.35$$

Consistency between Eqs. (14) and (15) requires

$$\alpha = c_e \tilde{k}/\tilde{\epsilon} \tag{16}$$

so that Eq.(13) becomes

$$h'' = -c_e \frac{\tilde{k}}{\tilde{\epsilon}} u_i'' \tilde{h}_{,i} \tag{17}$$

A more tensorially complex form of Eq. (13) could have been adopted to be consistent with the more recent, and empirically justified forms of Ref. 15. However, since the current work of introducing the effects of compressibility into the k and ϵ equations will later involve thin layer approximations which force the principal fluxes of Refs.14 and 15 to be identical, the simpler form of Eq. (13) will be used here.

The experimental data of Ref. 13 can be used to test the validity of Eq. (17) provided the differences in the physical assumptions and dependent variables used in the data reduction and this analysis can be reconciled. The data are expressed in unweighted ensemble averaged quantities whereas mass weighted variables are used here. The quantities measured in the experiment through the use of a hot wire with a large range of overheat values were the fluctuations of total temperature and axial mass flow. These quantities were then converted to static temperature and velocity fluctuations through the assumptions in Ref. 13 that static pressure fluctuations and higher order correlations could be neglected. With the latter assumption, the distinction between mass weighted and unweighted ensemble averaged quantities vanishes. Consequently, a direct comparison between the results of Ref. 13 and Eq. (17) is possible.

In a boundary layer, the principal terms on the right side of Eq. (17) are

$$h'' = -c_e \frac{\tilde{k}}{\tilde{\epsilon}} u_2'' \tilde{h}_{,2} \tag{18}$$

Similarly, the principal shear stress in the boundary layer, given by two equation modeling, is

$$\tau_{12} = C_\mu \bar{\rho} (\tilde{k}^2/\tilde{\epsilon}) \tilde{u}_{1,2} \tag{19}$$

When Eq. (19) is used to eliminate $\tilde{\epsilon}$ in Eq. (18), the variances of the static enthalpy and the velocity can be related as

$$\frac{< h'' > / \tilde{h}}{< u_1'' > / \tilde{u}_1} = \left| \frac{\frac{c_e}{C_\mu} \frac{(\tau_{12}/\bar{\rho})}{\tilde{k}} \frac{< u''_2 >}{< u''_1 >} (\tilde{u}_1/\tilde{u}_{1\epsilon})}{(\tilde{h}/\tilde{h}_\epsilon)} \frac{\partial(\tilde{h}/\tilde{h}_\epsilon)}{\partial(\tilde{u}_1/\tilde{u}_{1\epsilon})} \right| \tag{20}$$

where the differentiation is performed at a fixed station.

In evaluating the right member of Eq. (20), the following values are employed:

$$c_\epsilon = 0.35$$

$$C_\mu = 0.09$$

$$\left(\frac{\tau_{12}}{\rho}\right)/\tilde{k} = 0.3$$

which is consistent with the above value of C_μ and the universal log law, and

$$< u_2'' > / < u_1'' > = 0.64$$

which corresponds to the simulated data of Ref. 16 at a y^+ of about 140. The mean enthalpy and velocity relationships are taken directly from the data of Ref. 13. The results from Eq. (20), which is based on Eq. (17), are compared with the corresponding experimental data of Ref. 13 in Fig. 1. In addition, Fig. 1 shows the relationship of the enthalpy, velocity variances that result from the use of Eq. (12) as evaluated in Ref. 6.

The use of Eq. (17) results in a ratio of the enthalpy and velocity variances that represents the hypersonic boundary layer experimental data of Ref. (13) reasonably well over most of the boundary layer. It is surprising that the computed results differ from the data as much as they do in the log region of the boundary layer, $y/\delta < 0.2$, where the assumptions of the anisotropies and shear stress relationships listed above whould have been expected to apply the best. In the region near the boundary layer edge, however, it is not surprising that the computed results are high because no allowance has been made for the shear stress to approach a zero value there. Nonetheless, these results based on Eq. (17) are a significant improvement over those that result from the use of Eq. (12), the basis of the work in Ref. 6. Accordingly, Eq. (17) is used in subsequent steps of this analysis. Note that Eq. (17) indicates that the sign of the effect on the enthalpy fluctuation caused by a velocity fluctuation depends on the sign of the local mean enthalpy gradient. Alternative directions of heat flux appear to have opposite effects on the turbulence kinetic energy.

Evaluation of $\overline{u_i''}$

From Eqs. (10) and (17)

$$\frac{\rho'}{\rho} = \frac{1}{(n-1)}\frac{\rho h''}{\bar{\rho}\tilde{h}} = -\frac{1}{(n-1)} c_\epsilon \frac{\rho}{\bar{\rho}}\frac{\tilde{k}}{\tilde{\epsilon}}\frac{\tilde{h}_{,j}}{\tilde{h}} u_j'' \tag{21}$$

which can be used with the definition of $\overline{u_i''}$ to yield

$$\overline{u_i''} = \frac{\overline{\rho' u_i''}}{\bar{\rho}} = \frac{1}{(n-1)}\frac{c_\epsilon}{\bar{\rho}}\frac{\tilde{k}}{\tilde{\epsilon}}\frac{\tilde{h}_{,j}}{\tilde{h}}\overline{\rho u_j'' u_i''} \tag{22}$$

Recall that

$$\tilde{h} = \frac{\tilde{a}^2}{(\gamma - 1)} \tag{23}$$

so that Eq. (22) can also be expressed as

$$\overline{u_i''} = \frac{(\gamma - 1)}{(n - 1)} c_\epsilon \frac{\tilde{k}}{\tilde{\epsilon}} \tilde{h}_{,j} \left[\frac{\overline{\rho u_i'' u_j''}}{\bar{\rho}\tilde{a}^2} \right] \tag{24}$$

The bracketed terms represent moments of turbulence Mach numbers, which vanish in incompressible flow, i. e. $\tilde{a} \to \infty$. Thus $\overline{u_i''}$ is a measure of the degree of compressibility of the turbulence. It is also interesting that $\overline{u_i''}$ has a value of zero in the absence of any local heat transfer, where $\tilde{h}_{,j} = 0$. The latter point can be seen directly when Eqs. (24) and (15) are combined to yield

$$\overline{u_i''} = -\frac{(\gamma - 1)}{(n - 1)} \frac{Q_i}{\bar{\rho}\tilde{a}^2} \tag{25}$$

In most aerodynamic applications, the principal mean pressure gradient is $\bar{p}_{,1}$, which identifies $\overline{u_1''}$ as the main quantity required in the last term on the right of the kinetic energy equation, Eq. (9). If the flow also behaves as a thin shear layer where $\tilde{h}_{,2}$ is much greater than $\tilde{h}_{,1}$ or $\tilde{h}_{,3}$, Eq. (24) reduces to

$$\overline{u_1''} = \frac{(\gamma - 1)}{(n - 1)} c_\epsilon \frac{\tilde{k}}{\tilde{\epsilon}} \tilde{h}_{,2} \left[\frac{\overline{\rho u_1'' u_2''}}{\bar{\rho}\tilde{a}^2} \right] \tag{26}$$

Since the Reynolds shear stress and the static enthalpy gradient in Eq. (26) can each possess different signs, $\overline{u_1''}$ can be either positive or negative. For example, within a planar, hot jet, $\tilde{h}_{,2}$ and $\overline{\rho u_1'' u_2''}$ change sign simultaneously on the centerline of the jet. Thus $\overline{u_1''}$ retains a negative sign on either side of the centerline. Alternatively, in a high speed boundary layer over a cooled surface, $\tilde{h}$ has a maximum value within the boundary layer, while the shear stress maintains the same sign everywhere. Here, the sign of $\overline{u_1''}$ is different below or above the point where $\tilde{h}$ is maximum and the $\tilde{k}$ equation will be affected differently in these different zones of the boundary layer. This may explain, in part, why attached boundary layers seem to exhibit less compressibility effects than do free shear layers at similar Mach numbers.

Evaluation of $\overline{p' u_{i,i}''}$ in the Kinetic Energy Equation

In terms of mass weighted variables, the continuity equation for the fluctuating quantities is

$$\rho'_{,t} + (\rho' \tilde{u}_j + \rho u_j'')_{,j} = 0 \tag{27}$$

After some algebraic manipulations and neglect of higher order terms, Eq. (27) becomes

$$(\overline{\rho'^2})_{,t} + \tilde{u}_j(\overline{\rho'^2})_{,j} + 2\overline{\rho'^2}\tilde{u}_{j,j} + 2\bar{\rho}_{,j}\overline{\rho' u''_j} + 2\bar{\rho}\overline{\rho' u''_{j,j}} = 0 \tag{28}$$

or

$$\overline{\rho' u''_{j,j}} = -\tilde{u}_{j,j}\frac{\overline{\rho'^2}}{\bar{\rho}} - \frac{\bar{\rho}_{,j}}{\bar{\rho}}\overline{\rho' u''_j} - \frac{1}{2\bar{\rho}}\left[(\overline{\rho'^2})_{,t} + \tilde{u}_{,j}(\overline{\rho'^2})_{,j}\right] \tag{29}$$

Finally, relating the pressure and density fluctuations as in Eq. (8), allows writing

$$\overline{p' u''_{j,j}} = \frac{n\bar{p}}{\bar{\rho}}\overline{\rho' u''_{j,j}} \tag{30}$$

or, with Eq. (28), the pressure rate of strain term becomes

$$\overline{p' u''_{j,j}} = n\bar{p}\left\{-\tilde{u}_{j,j}\frac{\overline{\rho'^2}}{\bar{\rho}^2} - \frac{\bar{\rho}_{,j}}{\bar{\rho}}\frac{\overline{\rho' u''_j}}{\bar{\rho}} - \frac{1}{2\bar{\rho}^2}\left[(\overline{\rho'^2})_{,t} + \tilde{u}_{,j}(\overline{\rho'^2})_{,j}\right]\right\} \tag{31}$$

Eq. (31) shows that the variance of the density fluctuations, $\beta = \sqrt{\overline{\rho'^2}}/\bar{\rho}$, plays a role in establishing the magnitude of the pressure, rate of strain term appropriate to the kinetic energy equation. In fact, when Eq. (31) is expressed directly in terms of the variance β, and use is made of the mean continuity equation and the definition of $\overline{u''_j}$, Eq. (22) simplifies to give

$$\overline{p' u''_{j,j}} = n\bar{p}\left\{\frac{\bar{\rho}_{,j}}{\bar{\rho}}\overline{u''_j} - \frac{1}{2}\left[(\beta^2)_{,t} + \tilde{u}_j(\beta^2)_{,j}\right]\right\} \tag{32}$$

Locally, the density variance can be established from Eqs. (10) and (17) as

$$\beta^2 = \left(\frac{c_\epsilon}{(n-1)\bar{\rho}\tilde{h}}\frac{\tilde{k}}{\tilde{\epsilon}}\right)^2 \left|\overline{\rho^2 u''_i u''_j}\tilde{h}_{,i}\tilde{h}_{,j}\right| \tag{33}$$

or with higher order terms dropped

$$\beta^2 = \left(\frac{c_\epsilon}{(n-1)\tilde{h}}\frac{\tilde{k}}{\tilde{\epsilon}}\right)^2 \frac{1}{\bar{\rho}}\left|\overline{\rho u''_i u''_j}\tilde{h}_{,i}\tilde{h}_{,j}\right| \tag{34}$$

In a thin shear layer, where $\tilde{h}_{,2}$ predominates, Eq. (34) reduces to

$$\beta^2 = \frac{c_\epsilon^2}{(n-1)^2\tilde{h}^2}\frac{\tilde{k}^3}{\tilde{\epsilon}^2}\left(\frac{\overline{\rho u''^2_2}}{\bar{\rho}\tilde{k}}\right)\left(\tilde{h}^2_{,2}\right) \tag{35}$$

In the computation of a typical k-ϵ model, each of the quantities appearing in Eq. (35) are known local quantities and can be used to establish the local field value of the density variance required in Eq. (32).

The remaining terms in Eq. (9) that require modeling are the transport or diffusion of kinetic energy

$$D_k = -(\overline{\rho u_j'' k''} + \overline{u_j'' p'} - \overline{u_i'' \tau_{ij}''})_{,j} \tag{36}$$

and the turbulence dissipation

$$\tilde{\epsilon} = \overline{u_{i,j}'' \tau_{ij}''} \tag{37}$$

The reasons for identifying the whole of Eq. (36) as a transport term is its divergence form and its appearance in the total enthalpy equation, Eq. (8), where the pressure fluctuation is an implicit part of the enthalpy fluctuation. The dissipation term, Eq. (37), however, does not occur in the total enthalpy equation since it merely represents an exchange between mechanical energy and heat, the sum of which contributes to the dependent variable of that equation. In incompressible flows, and in Ref. (11), these terms are combined and their difference is modeled. Here, however, these terms will be treated separately because their molecular terms are important in different parts of the boundary layer and this can be used to advantage in the turbulence modeling of the additional compressibility terms.

The turbulent parts of D_k are modeled to be consistent with the level of modeling in Eq. (15), namely

$$\overline{\rho u_j'' k''} + \overline{u_j'' p'} = -c_s \frac{\tilde{k}}{\tilde{\epsilon}} \overline{\rho u_j'' u_i''} \tilde{k}_{,i} \tag{38}$$

where $c_s = 0.25$, Ref. 17.

For a thin shear layer, the principal direction of gradients in the kinetic energy require $i = j = 2$, so that Eq. (38) reduces to

$$\overline{\rho u_2'' k''} + \overline{u_2'' p'} = -c_s \frac{\tilde{k}}{\tilde{\epsilon}} \overline{\rho u_2'' u_2''} \tilde{k}_{,2} \tag{39}$$

With $c_s = 0.25$ and $\overline{\rho u_2'' u_2''}/\bar{\rho}\tilde{k} \simeq 0.36$, Eq. (29) can be rewritten as

$$\overline{\rho u_2'' k''} + \overline{u_2'' p'} = -.09 \bar{\rho} \frac{\tilde{k}^2}{\tilde{\epsilon}} \tilde{k}_{,2} \tag{40}$$

which is consistent with the simple gradient diffusion form of the standard $k - \epsilon$ model formulation.

The viscous part of Eq. (36) is

$$(\overline{u_i'' \tau_{ij}''})_{,j} = \left[\overline{\tilde{\mu} u_i'' (u_{i,j}'' + u_{j,i}'' - \frac{2}{3} \delta_{ij} u_{k,k}'')} \right]_{,j} \tag{41}$$

where the effects of the fluctuating part of the viscosity have been neglected. Because these viscous terms are only important in comparison with their turbulent counterparts very close to the surface where the turbulence has been damped, the extra compressibility effects resulting from mass weighting are not expected to be important where the viscous terms dominate. The turbulence, therefore, in these terms can be treated as solenoidal and $\overline{u_i''u_i''}/2 = \tilde{k}$. Eq. (41) then becomes

$$(\overline{u_i''\tau_{ij}''})_{,j} = \left[\bar{\mu}(\tilde{k}_{,j} + (\overline{u_i''u_j''})_{,i})\right]_{,j} \tag{42}$$

In incompressible flows, when the diffusion and dissipation terms are combined, only the first term on the right of Eq. (42) remains to define the viscous diffusion of the turbulence kinetic energy. This turns out to be a reasonable approximation even in inhomogeneous compressible turbulent flows. In the near wall region, the principal gradients of the turbulence moments are normal to the surface, where $i = j = 2$. At the surface, $\tilde{k} \sim y^2$, whereas $\overline{u_2''u_2''} \sim y^4$, so that the latter can be neglected. The second term becomes a larger contributor with distance from the surface, however, there the contributions of both terms in Eq. (42) become less significant in comparison with the turbulent transport. It has been estimated with simple mixing length arguments that the second term in Eq. (42) increases the local transport of kinetic energy by a maximum of 13 percent at a value of $y^+ = 12$. Over the entire sublayer, $0 < y^+ < 60$, this term increases the diffusion of $\tilde{k}$ only by about 4 percent, so that Eq. (42) can be simplified to

$$(\overline{u_i''\tau_{ij}''})_{,j} = (\bar{\mu}\tilde{k}_{,j})_{,j} \tag{43}$$

which can be used in both Eqs. (8) and (9).

With Eqs. (38) and (43)

$$D_k = \left[c_s\frac{\tilde{k}}{\tilde{\epsilon}}\overline{\rho u_j''u_i''}\tilde{k}_{,i} + \bar{\mu}\tilde{k}_{,j}\right]_{,j} \tag{44}$$

or in a thin layer

$$D_k = \left[(c_s\frac{\tilde{k}}{\tilde{\epsilon}}\overline{\rho u_2''u_2''} + \bar{\mu})\tilde{k}_{,2}\right]_{,2} \tag{45}$$

The dissipation term, Eq. (37), expands to

$$\begin{aligned}\epsilon &= \overline{\bar{\mu}u_{i,j}''(u_{i,j}'' + u_{j,i}'' - \frac{2}{3}\delta_{ij}u_{k,k}'')}\\ &= \bar{\mu}\overline{u_{i,j}''u_{i,j}''} + \bar{\mu}\overline{u_{i,j}''u_{j,i}''} - \frac{2}{3}\bar{\mu}\overline{(u_{k,k}'')^2}\end{aligned} \tag{46}$$

It is noted that the "dilitation dissipation", defined in Ref. 11 and modeled there, appears as the third term on the right of Eq. (46). Here, to account for inhomogeneous turbulence, the dissipation will be treated as consisting of all three terms in Eq. (46) and its difference from the first term, the ϵ of the standard $k - \epsilon$ model, will require modification of the form of the ϵ equation.

Those readers familiar with the details of Ref. 11 will note that the "dilitation dissipation" term above has a different sign and coefficient from that of Ref. 11. The reason for this is that the form of the total dissipation of Ref. 11 resulted from manipulation of the second term of Eq. (46) under the assumption that the fluctuating turbulent velocity moments, not necessarily the Reynolds stresses which include local density, are homogeneous. It is interesting that an identical result occurs if the diffusion and dissipation terms are first combined, but then it is required to assume that both the velocity moments and the viscosity are homogeneous. The approach used in this paper avoids these restrictions, and allows for both inhomogeneous turbulence and physical properties that are variable.

In summary, the $k - \epsilon$ model has been modified for the additional effects of compressibility in the following manner. The kinetic energy equation, Eq. (9), is rewritten as

$$(\bar{\rho}\tilde{k})_{,t} + (\bar{\rho}\tilde{u}_j\tilde{k})_{,j} = -(\overline{\rho u_i'' u_j''})\tilde{u}_{i,j} - \bar{\rho}\tilde{\epsilon} + D_k + \overline{p' u''_{i,i}} - \overline{u_i''}\bar{p}_{,i} \tag{47}$$

where D_k is given by Eq. (44), $\overline{p' u''_{i,i}}$ is given by Eqs. (32) and (34) or (35), and $\overline{u_i''}$ is given by Eq. (24) or Eq. (26). The dissipation, $\tilde{\epsilon}$, is modeled with the field equation

$$(\bar{\rho}\tilde{\epsilon})_{,t} + (\bar{\rho}\tilde{u}_j\tilde{\epsilon})_{,j} = -C_{\epsilon 1}\frac{\tilde{\epsilon}}{\tilde{k}}(\overline{\rho u_i'' u_j''})\tilde{u}_{i,j} - C_{\epsilon 2}\bar{\rho}\frac{\tilde{\epsilon}^2}{\tilde{k}} + D_\epsilon + C_{\epsilon 3}\frac{\tilde{\epsilon}}{\tilde{k}}\overline{p' u''_{i,i}}$$

$$-C_{\epsilon 4}\frac{\tilde{\epsilon}}{\tilde{k}}\overline{u_i''}\bar{p}_{,i} - \bar{\rho}\tilde{\epsilon}\tilde{u}_{j,j} \tag{48}$$

where

$$D_\epsilon = (c_\epsilon \frac{\tilde{k}}{\tilde{\epsilon}}(\overline{\rho u_i'' u_j''})\tilde{\epsilon}_{,i} + \tilde{\mu}\tilde{\epsilon}_{,j})_{,j} \tag{49}$$

To most clearly illustrate the additional terms introduced to account for the effects of compressibility, equations (47) and (48) are given in their high Reynolds number forms, absent any near wall corrections. Eq. (48) contains three terms in addition to those contained in the ϵ equation of the standard k-ϵ model. The third term on the right has been added to account for the expected dependence of turbulence length scale on passing through a shock wave following arguments similar to those of Refs. (9); it does not contain the factor of 1/3 recommended in Ref. (10) because the latter is only appropriate to isotropic turbulence. This term is necessary even when the effects of fluctuations in density are ignored. The other two terms, those containing $C_{\epsilon 3}$ and $C_{\epsilon 4}$, account for the effects of the fluctuating density on altering the total dissipation of kinetic energy. In addition to the polytropic coefficient, n, the two new modeling constants introduced are the $C_{\epsilon 3}$ and $C_{\epsilon 4}$. All the other modeling constants and relationships between $\tilde{k}$, $\tilde{\epsilon}$, the eddy viscosity, etc. are the same as in the usual incompressible k-ϵ approaches.

CONCLUDING REMARKS

The forgoing anlysis to define the extra compressibility terms that are introduced by fluctuating density in the Favre averaging process has resulted in the introduction of three new modeling constants: the poytropic coefficient, n, and two coefficients in the modified kinetic energy dissipation equation, $C_{\epsilon 3}$ and $C_{\epsilon 4}$. In the absence of validation computations that include these terms for experimental conditions where they can be important, it isn't possible at this time to give recommendations for their appropriate values. In Ref. 6, it was recommended that n=1.2. The basis for this recommendation was no more profound than that n=1.2 is consistent with polytropic irreversable expansion relationships in thermodynamic cycles often cited in standard thermodynamic text books,. e.g. Ref. 18. On the other hand, when the behavior accross a normal shock wave is considered to be polytropic, it is found that $n > \gamma$, the isentropic coefficient. In view of these differences and in the absence of any empirical guidance, it is believed that starting with n=1.4 is reasonable. It is the intent of the author to establish the value of $C_{\epsilon 3}$ first, through comparisons of computations with experimental data for flow fields involving zero mean pressure gradients, thereby eliminating the term containing $C_{\epsilon 4}$. Candidate flows for this are single or double stream free-shear layers and attached flat plate boundary layers, all in high speed flows. The effects of the added terms are expected to be less for the attached flow. Once consistent values of n and $C_{\epsilon 3}$ are established, test cases involving strong pressure gradients will be pursued to establish $C_{\epsilon 4}$.

REFERENCES

1. Favre, A.,"Equations des Gaz Turbulents Compressibles," J. Mecan., Vol. 4, No. 3, Sept. 1965, pp. 361-390.

2. Rubesin, M. W. and Rose, W. C., "The Turbulent Mean-Flow, Reynolds-Stress, and Heat-Flux Equations in Mass-Averaged Dependent Variables," NASA TM X-62,248, March 1973.

3. Varma, V. K., Beddini, R. A., Sullivan, R. D., and Donaldson, C. duP., "Application of Second-Order Closure Model to Compressible Turbulent Shear Layers," AIAA Paper 74-592, AIAA 7th Fluid and Plasma Dynamics Conference,June 17-19, 1974, Palo Alto, Calif.

4. Wilcox, D. C. and Alber, I. E., "A Turbulence Model for High Speed Flows," pp. 231-252, Proceedings of the 1972 Heat Transfer and Fluid Mechanics Institute, San Fernando Valley State College, June 14-16, 1972, Northridge, Calif.

5. Oh, Y. H., "Analysis of Two-Dimensional Free Turbulent Mixing," AIAA Paper No. 74-594, AIAA 7th Fluid and Plasma Dynamics Conference,June 17-19, 1974, Palo Alto, Calif.

6. Rubesin, M. W., "A One-Equation Model of Turbulence for Use with the Compressible Navier-Stokes Equations," NASA TM X-73,128, April 1976.

7. Vandromme, D., "Contribution a la Prediction et la Modelization d'Ecoulements Turbulents a Masse Volumique Variable," Sc.D. Thesis, University of Sciences and Techniques of Lille, France, 1983.

8. Reynolds, W. C.,"Modeling of Fluid Motions in Engines:

An Introductory Overview," Symposium on Combustion Modeling in Recriprocating Engines, Plenum Press, 1980, pp. 131-155.

9. Morel, T. and Mansour, N. N.,"Modeling of Turbulence in Internal Combustion Engines," SAE Technical Paper Series, 820040, International Congress and Exposition, Detroit, Mich., February 22-26, 1982.

10. El Tahry, S. H., "k-ϵ Equation for Compressible Reciprocating Engine Flows," J. Energy, Vol. 7, No. 4, July-August 1983.

11. Zeman, O., "Dilitation Dissipation: The Concept and Application in Modeling Compressible Mixing Layers," Physics of Fluids A, Fluid Dynamics, Vol. 2, No. 2, pp. 178-188, February 1990.

12. Papamoshou, D. and Roshko, A., "The Compressible Turbulent Shear Layer: An Experimental Study", J. Fluid Mech, Vol. 197,pp. 453-477, 1988.

13. Owen, F. K., Horstman, C. C., and Kussoy, M. I., "Mean and fluctuating flow meansurements of a fully-developed, non-adiabatic, hypersonic boundary layer," J. Fluid Mech., Vol. 70, Part 2, pp. 393-413, 1975.

14. Daly, B. J. and Harlow, F. H., "Transport Equations in Turbulence," Physics of Fluids, Vol. 13, No. 11, pp. 2634-2649, November, 1970.

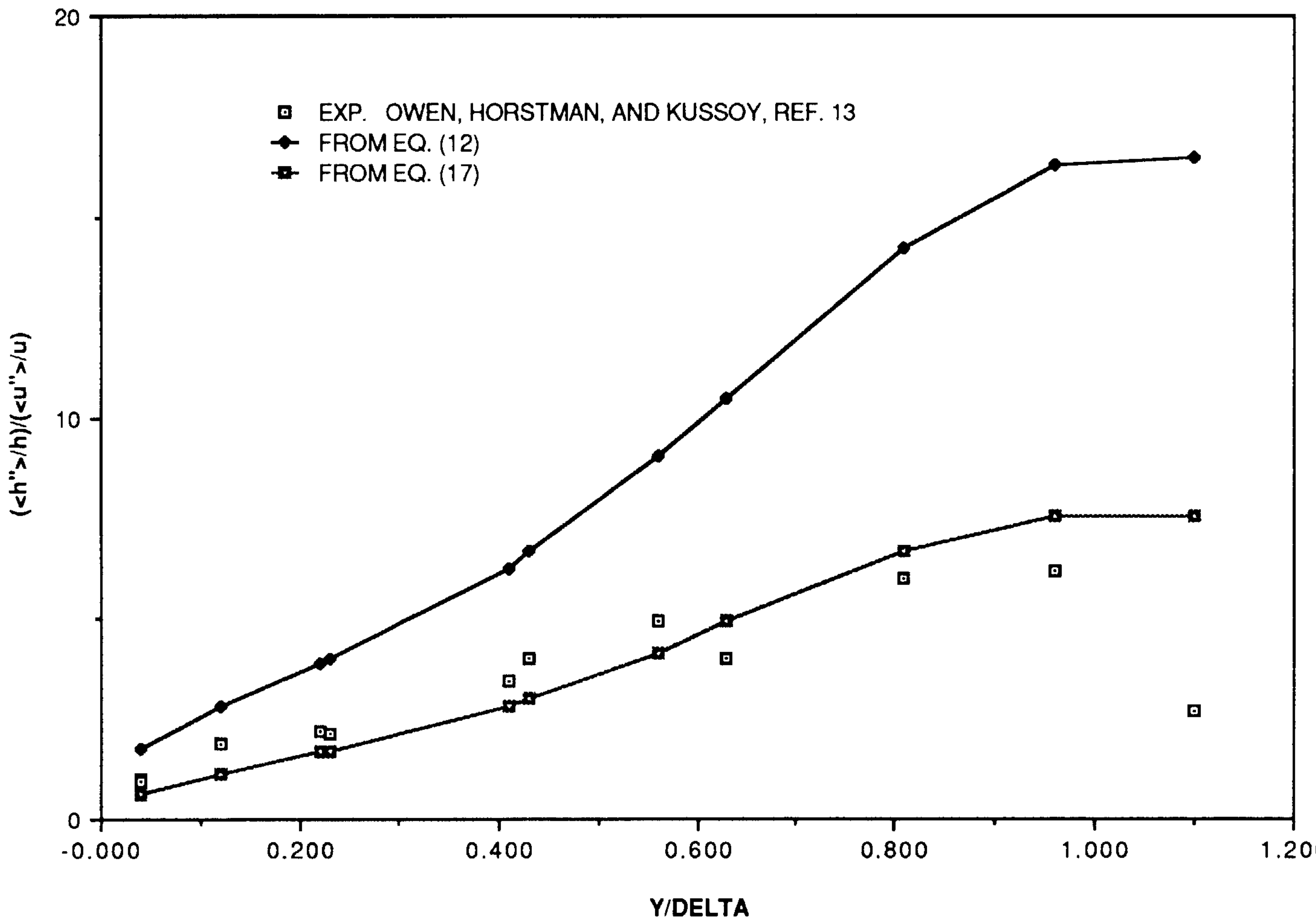
15. Rogers, M. M., Mansour, N. N., and Reynolds, W. C., "An algebraic model for the turbulent flux of a passive scalar," J. Fluid Mech., Vol. 203, pp. 77-101, 1989.

16. Moin, P. and Kim, J., "Numerical investigation of turbulent channel flow," J. Fluid Mech., Vol. 118, pp. 341-377, 1982.

17. Launder, B. E., Reece, G. J., and Rodi, W., "Progress in the Development of a Reynolds Stress Turbulence Closure," J. Fluid Mech., Vol. 68, Pt. 3, 1975, pp. 537-566.

18. Marks, J. S. (Ed.), *Mechanical Engineers' Handbook*, McGraw-Hill Book Co., New York, 1951, p. 295.

Fig. 1 RATIO OF ENTHALPY TO VELOCITY VARIANCES



NASA
National Aeronautics and Space Administration

Report Documentation Page

1. Report No.
NASA CR-177556

2. Government Accession No.

3. Recipient's Catalog No.

4. Title and Subtitle
Extra Compressibility Terms for Favre-Averaged Two-Equation Models of Inhomogeneous Turbulence

5. Report Date
June 1990

6. Performing Organization Code

7. Author(s)
Morris W. Rubesin

8. Performing Organization Report No.
A-90196

10. Work Unit No.
505-80-11

9. Performing Organization Name and Address
MCAT Institute
Ames Research Center
Moffett Field, CA 94035-1000

11. Contract or Grant No.
NCC2-585

13. Type of Report and Period Covered
Contractor Report

12. Sponsoring Agency Name and Address
National Aeronautics and Space Administration
Washington, DC 20546-0001

14. Sponsoring Agency Code

15. Supplementary Notes
Point of Contact: John R. Viegas, Ames Research Center, M/S 229-1, Moffett Field, CA 94035-1000
(415) 604-5950 or FTS 464-5950

16. Abstract

Forms of extra compressibility terms that result from use of Favre averaging of the turbulence transport equations for kinetic energy and dissipation are derived. These forms introduce three new modeling constants, a polytropic coefficient that defines the interelationships of the pressure, density, and enthalpy fluctuations and two constants in the dissipation equation that account for the non-zero pressure-dilitation and mean pressure gradients.

17. Key Words (Suggested by Author(s))
Turbulence modeling
Compressibility
Favre averaging

18. Distribution Statement
Unclassified–Unlimited

Subject Category–34

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	20	A02

NASA FORM 1626 OCT 86

For sale by the National Technical Information Service, Springfield, Virginia 22161